Learning Enhancement Team

University of East Anglia

STUDENT SUPPORT SERVICE

Steps into Discrete Mathematics

What are Determinants and Inverses of Matrices?

This guide introduces determinants and inverse matrices and shows how the two are related through both algebra and geometry.

Introduction

If a matrix has the same number of rows as it does columns then it is called a **square matrix**. Each square matrix has a number associated with it called a **determinant** and this number is used in calculating (if it is possible to calculate it) the **inverse** of that matrix. Inverse matrices are crucial in the study of matrices (also known as **linear algebra**) and are used to solve large systems of simultaneous equations in algebra, **optimization** and **computational mathematics**.

If you are not familiar with the basic terminology of matrices, then you can read the study guide: *Basics of Matrices*. This guide also uses matrix multiplication, so please read the study guide: *Multiplying Matrices* if you are unfamiliar with this. This study guide does not describe how to compute determinants or inverses. For that you can read the study guides: *Calculating Determinants* and *Calculating Inverses*. This study guide will help you understand what determinants and inverses are and how they are related geometrically and algebraically. In particular you will learn that every square matrix has a determinant but not every square matrix has an inverse. Specifically:

> If the determinant of a matrix is zero then the matrix does not have an inverse.

Notation

In this study guide all matrices will be assumed to be square and written as an upper case letter such as *A.*

> The determinant of a square matrix A is written as: $\det(A)$ or $|A|$.

This guide will use $\det(A)$.

If $\det(A) \neq 0$ then the matrix A has an inverse.

> The inverse of a matrix A is written as: A^{-1}.

Here, the superscript -1 means it is an inverse and that not that A is raised to the power of -1. You will see the reason for this notation later.

If a matrix A has $\det(A) = 0$ then it is called a **singular** matrix and a singular matrix **does not have an inverse**.

What is an inverse matrix?

Multiplication by the inverse matrix A^{-1} "undoes" multiplication by A. If you multiply a matrix B by A then you will have AB. If you then multiply by A^{-1} you will undo the multiplication by A to get back to B. In other words:

$$A^{-1}AB = B$$

The only matrix which, when it multiplies, does not change things is the identity matrix I (see the study guide: *Basics of Matrices*) and so multiplying by A and then by A^{-1} must be the same as multiplying by I and so:

$$\boxed{AA^{-1} = I}$$

The same is true if you multiply by A^{-1} first and so $A^{-1}A = I$ too. It does not matter which order you multiply A and A^{-1} in although, in general, AB does not always equal BA. In other words, matrices and their inverses (if they exist) commute but in general matrix multiplication is not commutative.

Solving equations

A good way of understanding what an inverse matrix is, is to consider how to solve simultaneous equations. In algebra you might want to solve the equation:

$$ax = b$$

where b and a are known numbers but a is not zero. To find x you would need to "undo" the multiplication on the left hand side by dividing by a or, in other words, to multiply by a^{-1} and you would get the solution $x = a^{-1}b$.

The same method can be used on simultaneous equations but you must first understand how simultaneous equations can be written as matrix equations.

Example: Solve the simultaneous equations $3x - 2y = 7$ and $x + y = 4$.

You could use the method described in the study guide: *Simultaneous Equations* to solve this but instead you can write this as a matrix equation:

$$\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

If you multiply these matrices out, you will obtain the simultaneous equations in the question. You can check this yourself.

Next, you need to find the inverse matrix. The study guide: *Finding Inverses* will show you how to do this but, for now, you can confirm that if the matrix:

$$\frac{1}{5}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \text{ is the inverse of the matrix } \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$$

then, left-multiplying this with the original matrix gives:

$$\frac{1}{5}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and so this is indeed the inverse. Therefore, multiplying both sides of the matrix equation above by this inverse gives:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 7 \\ 4 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 15 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

and so the solution is $x = 3$ and $y = 1$ and you can check this by putting these values into the equations in the question.

In fact, any set of two linear simultaneous equations can be written in this form:

$$Ax = b$$

In the example above:

$$\underset{A}{\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}}\underset{x}{\begin{bmatrix} x \\ y \end{bmatrix}} \underset{=}{=} \underset{b}{\begin{bmatrix} 7 \\ 4 \end{bmatrix}}$$

A is the known 2×2 matrix, *x* is the unknown 2×1 matrix and *b* is the known 2×1 matrix. To solve this you must multiply both sides of the equation from the left by the inverse matrix A^{-1} so that you get:

$$A^{-1}Ax = A^{-1}b$$

Just as with the simple algebra problem above, multiplying on the left by A^{-1} "undoes" multiplying by *A* and this allows you to solve for *x*:

$$x = A^{-1}b$$

So, if you know A^{-1}, you just need to do this matrix multiplication to solve for *x.*

In the example above, the determinant of the matrix in the question is 5 and that is why the fraction 1/5 multiplies the inverse. Now you can see that, if the determinant were 0, there would be no solution to the system because you cannot divide by 0.

Any system of *n* linear simultaneous equations can be described by a matrix equation $Ax = b$ where *A* is a known $n \times n$ matrix and *x and b* are $n \times 1$ matrices.

> If $\det(A) \neq 0$ then the set of equations has a unique solution.
>
> If $\det(A) = 0$ then the set of equations has no unique solution.

Example: Solve the simultaneous equations $-2x - 2y = 7$ and $x + y = 4$.

These equations can be written as the matrix equation:

$$\begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

The determinant of $\begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix}$ is 0 and so there is no inverse for this matrix and therefore no solution to this system. You can try to solve the simultaneous equations using a standard method to confirm this.

At the start of this section you considered the scalar equation $ax = b$ where a cannot be zero because this would not have a solution. For matrix equations $Ax = b$, the equivalent condition is that $\det(A) \neq 0$ because:

> If $\det(A) = 0$ then *A* has no inverse.

Geometric interpretation of determinants and inverses

The square in the figure below has four corners with coordinates $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$ and is known as the **unit square** since the length of its side is 1 and also it has an area of 1. These coordinates can be written in the matrix *S* (for "square"):

$$S = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

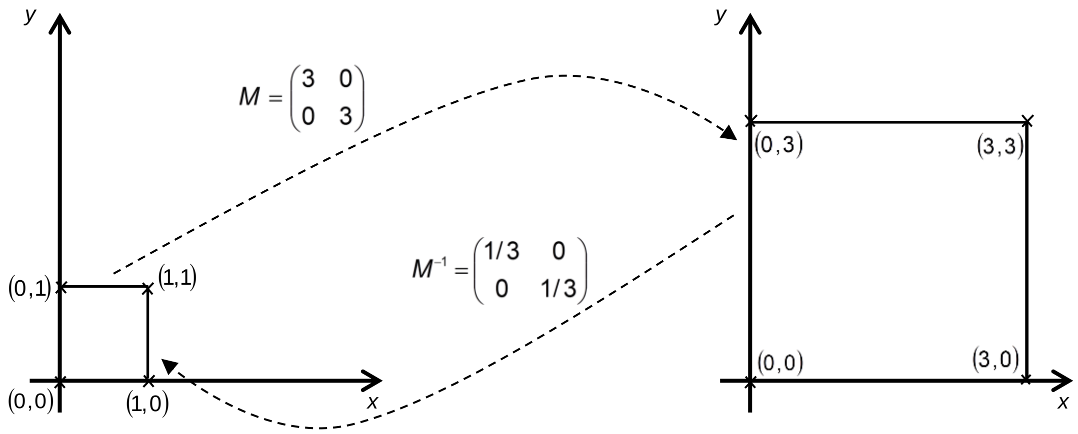
When each of these coordinates are multiplied from the left by the diagonal matrix:

$$M = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

the coordinates become $(0,0)$, $(0,3)$, $(3,0)$ and $(3,3)$ because:

$$MS = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 3 & 3 \\ 0 & 3 & 0 & 3 \end{bmatrix}$$

which is the square in the figure on the right and which has area 9.



If these new coordinates are then multiplied from the left by the diagonal matrix:

$$M^{-1} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix}$$

then they become the coordinates of the unit square again because:

$$M^{-1}MS = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 3 & 3 \\ 0 & 3 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

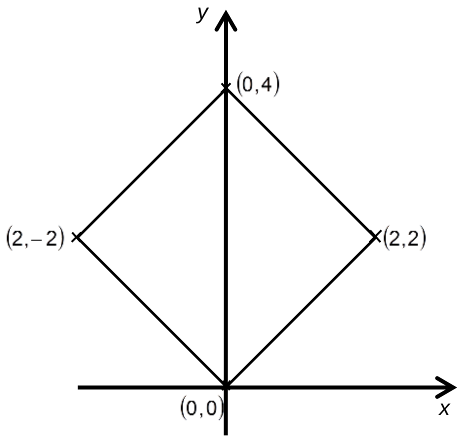
and this is because the matrix M^{-1} is the inverse of *M.*

The determinant of *M* is 9 and the determinant of M^{-1} is $1/9$. The matrix *M* increases the area of a shape by a factor of 9 and the matrix M^{-1} decreases the area of any shape by a factor of 9 and so, in 2D, the geometric interpretation of determinant of a 2×2 matrix is that it describes the change in area. Similarly, in 3D, the geometric interpretation of determinant of a 3×3 matrix is that it describes the change in volume.

Example: What is the change in area of the unit square when it is multiplied by the matrix $M = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$?

If you multiply the matrix of the coordinates of the unit square *S* by *M*, you get:

$$MS = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 2 & 0 \\ 0 & 2 & 2 & 4 \end{bmatrix}$$



Looking at the figure on the left, you can use Pythagoras' theorem to find that the length of the side of this square is $\sqrt{8}$ and so the area of the square is 8. The square's area has increased by a factor of 8 and so the determinant is 8.

If the unit square was multiplied by the null matrix: $N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ then all the coordinates of the unit square or of any shape will all transform to the origin or $(0,0)$. The area of the shape will have decreased to 0 and so $\det(N) = 0$. There is no way to undo this operation with a matrix since, whatever matrix the coordinate $(0,0)$ is multiplied by, it will still be $(0,0)$. No matter how much the area is scaled up it will still be zero. Because $\det(N) = 0$, there is no inverse for *N*.

Want to know more?

If you have any further questions about this topic you can make an appointment to see a **Learning Enhancement Tutor** in the **Student Support Service**, as well as speaking to your lecturer or adviser.

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