## Learning Enhancement Team

### *Steps into Discrete Mathematics*

# Calculating Determinants

***This guide describes how to calculate determinants of 2x2 and 3x3 matrices. It also goes into more depth about how the method for finding the determinant of a 3x3 matrix is derived.***

## Introduction

Every **square matrix** has a number associated with it called a **determinant**. The determinant of a matrix $A$ is written as either $\det(A)$ or $|A|$, this guide will use $\det(A)$. It is extremely useful to know how to calculate determinants and this guide concentrates on methods to do so for $2\times 2$ and $3\times 3$ matrices. It also explores how a quick formula for calculating the determinant of a $3\times 3$ matrix is derived. If you are unfamiliar with what a matrix is you can learn more from the study guide: *Basics of Matrices*.

Determinants are extremely important in linear algebra, the geometric and algebraic significance of determinants is described in the study guide: *What are Determinants and Inverses?*. The calculation of determinants proves useful in other areas of mathematics such as working out the **cross product** of two vectors (see study guide: *The Cross Product*), the **curl** of a vector field (see study guide: *Curl*) and the transformation of one coordinate system to another. A **Learning Enhancement Tutor** would be happy to discuss these advanced topics with you in more detail.

## Determinant of a 2×2 matrix

If a $2\times 2$ matrix $A$ is given by:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The determinant is given by:

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad \text{determinant of a } 2\times 2 \text{ matrix}$$

You should notice that the **elements** of the matrix are enclosed in **square brackets** and the **determinant** is enclosed in **vertical lines**, this is an important piece of notation and you should make sure you use it correctly.

*Example*: What is the determinant of $A=\begin{bmatrix}-2 & 1\\ 5 & 7\end{bmatrix}$?

Using the formula above:

$$\det(A)=\begin{vmatrix}-2 & 1\\ 5 & 7\end{vmatrix}=(-2)\cdot 7-1\cdot 5=-14+5=-9$$

So the determinant of this matrix is $-9$.

## General formula for a 3x3 determinant

There is a formula which calculates the determinant of a matrix $3\times 3$.

For $A=\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}$

$$\det(A)=\begin{vmatrix}a & b & c\\ d & e & f\\ g & h & i\end{vmatrix}=a(ei-fh)-b(di-fg)+c(dh-eg)$$

Determinant of a $3\times 3$ matrix

You can use this formula by substituting the elements from the matrix in directly.

*Example*: What is the determinant of $A=\begin{bmatrix}4 & 3 & -2\\ 2 & 5 & 0\\ 1 & -1 & -3\end{bmatrix}$?

Substituting the values of *A* into the formula gives, using a dot to denote multiplication:

$$\det(A)=\begin{vmatrix}4 & 3 & -2\\ 2 & 5 & 0\\ 1 & -1 & -3\end{vmatrix}=4(5\cdot(-3)-0\cdot(-1))-3(2\cdot(-3)-0\cdot 1)+(-2)(2\cdot(-1)-5\cdot 1)=-28$$

So the determinant of *A* is $-28$.

## Deriving the determinant of a 3×3 matrix: blocking out

The formula for finding the determinant of a $3\times 3$ matrix is more complicated than that for the a $2\times 2$ matrix but it does use the latter result its derivation. The derivation involves **blocking out** the matrix. Let's use matrix *A* from the previous example to help understand the process.

**Step 1: Choose a row or column of the matrix.**

To begin blocking out you chose any row or column in the matrix. A special property of determinants is that it does not matter which row or column you choose as you will always get the same result. The top row of the matrix is the most common one to pick so this example will use that.

**Step 2: *Block out* the row and column that the first element is in.**

Here the first element in the top row is 4 so you would *block out* the top row and first column like this:

$$A = \begin{bmatrix} 4 & 3 & -2 \\ 2 & 5 & 0 \\ 1 & -1 & -3 \end{bmatrix} \quad \text{this reveals a } \textbf{submatrix} \quad \begin{bmatrix} 5 & 0 \\ -1 & -3 \end{bmatrix}$$

**…and then calculate the determinant of the resulting submatrix** using the formula for a $2 \times 2$ matrix from earlier in this guide.

$$\begin{vmatrix} 5 & 0 \\ -1 & -3 \end{vmatrix} = 5 \cdot (-3) - 0 \cdot (-1) = -15$$

In mathematics this is called calculating the **minor** of the first element. A minor of an element in row *i* and column *j* is given the symbol $M_{ij}$. So, as the element is in row 1 and column 1, the minor is $M_{11} = -15$.

**Step 3: Adjust the sign of the number you have just calculated if necessary.**

You now have to multiply your result by either 1 or $-1$, depending on where the element you are blocking out is positioned in the original matrix. The pattern for a $3 \times 3$ matrix is:

$$\begin{bmatrix} +1 & -1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \end{bmatrix}$$

As you can see you alternate between multiplying by $+1$ and $-1$. Here you are using the top left element and so you multiply the minor by $+1$. In mathematics this result is called the **cofactor** $C_{ij}$ where *i* and *j* are the same as for the minor in step 2. So the cofactor of the first element is $C_{11} = (-15) \cdot (+1) = -15$.

**Step 4: Multiply the cofactor by the element you originally picked to give its contribution to the determinant.**

Here the original element is 4 and so you end up with:

$$4C_{11} = -60$$

And this is the contribution to the determinant from this element.

**Step 5:** **Repeat steps 2 to 4 for the other elements in the row or column you chose.** The second element of the top row is 3 so you block out the top row and second column:

$$A=\begin{bmatrix}4 & 3 & -2\\ 2 & 5 & 0\\ 1 & -1 & -3\end{bmatrix} \quad \text{this reveals the } \textbf{submatrix} \quad \begin{bmatrix}2 & 0\\ 1 & -3\end{bmatrix}$$

So the minor is: $M_{12}=\begin{vmatrix}2 & 0\\ 1 & -3\end{vmatrix}=2\cdot(-3)-0\cdot1=-6$,

the cofactor is: $C_{12}=(-1)\cdot(-6)=6$

and the contribution is: $3C_{12}=18$

The third element of the top row is $-2$ so you block out the top row and second column:

$$A=\begin{bmatrix}4 & 3 & -2\\ 2 & 5 & 0\\ 1 & -1 & -3\end{bmatrix} \quad \text{this reveals the } \textbf{submatrix} \quad \begin{bmatrix}2 & 5\\ 1 & -1\end{bmatrix}$$

So the minor is: $M_{13}=\begin{vmatrix}2 & 5\\ 1 & -1\end{vmatrix}=2\cdot(-1)-5\cdot1=-7$,

the cofactor is: $C_{13}=1\cdot(-7)=-7$

and the contribution is: $-2C_{13}=14$

**Step 6:** **Add up the contributions to find the determinant.** You can write this as:

$$\det(A)=\begin{vmatrix}4 & 3 & -2\\ 2 & 5 & 0\\ 1 & -1 & -3\end{vmatrix}=4\begin{vmatrix}5 & 0\\ -1 & -3\end{vmatrix}-3\begin{vmatrix}2 & 0\\ 1 & -3\end{vmatrix}-2\begin{vmatrix}2 & 5\\ 1 & -1\end{vmatrix}=-60+18+14=-28$$

So $\det(A)=-28$.

As you can see this is a long and tricky calculation. You must take care when performing it, especially when dealing with matrices with minus signs in them. However there are some choices you can make which may help shorten the calculation. For example **if any row or column has zeroes in it then chose this as the basis for you calculation** remember that:

> The determinant of a matrix is the same regardless of the row or column you chose.

*Example*: Use the second row of matrix $A$ to find $\det(A)$.

**Step 1: Choose a row or column of the matrix.** The example says use the second row.

**Step 2: Block out the row and column that the first element is in and then calculate the determinant of the resulting submatrix.**
The first element of the second row is 2 so block out the second row and first column:

$$A = \begin{bmatrix} 4 & 3 & -2 \\ 2 & 5 & 0 \\ 1 & -1 & -3 \end{bmatrix} \quad \text{this reveals a } \textbf{submatrix} \quad \begin{bmatrix} 3 & -2 \\ -1 & -3 \end{bmatrix}$$

So $\begin{vmatrix} 3 & -2 \\ -1 & -3 \end{vmatrix} = 3 \cdot (-3) - (-2) \cdot (-1) = -11$

This is the **minor** of the first element: $M_{21} = -11$.

**Step 3: Adjust the sign of the number you have just calculated if necessary.** Using:

$$\begin{bmatrix} +1 & -1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \end{bmatrix}$$

you multiply the minor by $-1$. So the cofactor $C_{21} = (-1) \cdot (-11) = 11$.

**Step 4: Multiply the cofactor by the element you originally picked to give its contribution to the determinant.** The original element from $A$ was 2 and so you end up with:

$2C_{21} = 22$

Giving the contribution to the determinant from this element.

**Step 5: Repeat steps 2 to 4 for the other elements in the row or column you chose.**
The second element of the second row is 3 so you block out the second row and second column:

$$A = \begin{bmatrix} 4 & 3 & -2 \\ 2 & 5 & 0 \\ 1 & -1 & -3 \end{bmatrix} \quad \text{this reveals the } \textbf{submatrix} \quad \begin{bmatrix} 4 & -2 \\ 1 & -3 \end{bmatrix}$$

So the minor is: $M_{22} = \begin{vmatrix} 4 & -2 \\ 1 & -3 \end{vmatrix} = 4 \cdot (-3) - (-2) \cdot 1 = -10$,

the cofactor is: $C_{12} = 1 \cdot (-10) = -10$

and the contribution is: $5C_{12} = -50$

The third element of the top row is 3 so you block out the top row and second column:

$$A = \begin{bmatrix} 4 & 3 & -2 \\ 2 & 5 & 0 \\ 1 & -1 & -3 \end{bmatrix} \qquad \text{this reveals the } \textbf{submatrix} \begin{bmatrix} 4 & 3 \\ 1 & -1 \end{bmatrix}$$

So the minor is: $M_{13} = \begin{vmatrix} 4 & 3 \\ 1 & -1 \end{vmatrix} = 4\cdot(-1) - 3\cdot 1 = -7$,

the cofactor is: $C_{23} = 0 \times (-7) = 0$

and the contribution is zero

**Step 6:** **Add up the contributions to find the determinant.** You can write this as:

$$\det(A) = \begin{vmatrix} 4 & 3 & -2 \\ 2 & 5 & 0 \\ 1 & -1 & -3 \end{vmatrix} = -50 + 22 + 0 = -28$$

So $\det(A) = -28$ which is the same answer as before. The advantage of picking the middle row is that that the final term is zero which reduces the number of calculations you have to do.

## Want to know more?

If you have any further questions about this topic you can make an appointment to see a **Learning Enhancement Tutor** in the **Student Support Service**, as well as speaking to your lecturer or adviser.


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