UEA University of East Anglia
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Steps into Discrete Mathematics

Basics of Logic

This guide introduces propositional logic and defines a proposition and the logical connectives "and", "or", "not", "conditional" and "biconditional" in terms of truth tables. It also looks at tautologies and contradictions.

Introduction

Logic comes in many forms and has played a central role in the development of language, philosophy and mathematics. This study guide will begin to explain the importance of logic for mathematics and especially computer science. When you study mathematics before university you generally learn tools to help you solve problems, at university the study of mathematics changes somewhat to focus on **proof**. Proof uses **mathematical logic** to determine the truth of statements and is used by mathematicians to construct **theorems** and test **conjectures** and **hypotheses**. The history of logic is long and rich with its foundations in Ancient Greece (Aristotle), through the British logicians (Boole, De Morgan) to 'modern' mathematical philosophers (Russel, Gödel). This study guide is about the type of mathematical logic that is concerned with the construction and analysis of **propositions** and is often called **propositional logic** (or sometimes **propositional calculus**). Propositional logic can be extended to other forms of logic such as predicate logic and first and second order logics.

Propositions: true or false?

In propositional logic a **proposition** (often called a **statement**) can be either true or false, but never both.

> A **proposition** (or **statement**) is either **true** or **false** but not both.

In general, instructions or questions are not propositions.

Example: Which of these are basic propositions:

(i) It is sunny. (ii) How are you?
(iii) Put that down! (iv) $8 \div 2 = 4$

(i) and (iv) are basic propositions as they can be true or false, (ii) is a question and so is not a proposition and (iii) is an instruction and so is also not a proposition.

In the study of logic, symbols are used to denote basic propositions like "It is sunny". It is conventional for an initial proposition to be denoted by *p* and any subsequent ones denoted by *q*, *r*, *s* and so on.

Basic logical operations: connectives

You can join basic propositions together to make longer logical propositions using **logical connectives**. These longer propositions are also true or false but not both.

> Any combination of propositions and logical connectives are also propositions.

There are five basic logical connectives that you should be familiar with: **conjunction** (logical "and"), **disjunction** (logical "or"), **negation** (logical "not"), **conditional** and **biconditional**. As you can see three of these connectives have common uses in language but remember that they have strict definitions in logic which are outlined below.

1. Conjunction

Two propositions can be joined together using a connective called **conjunction** it is given the symbol $\wedge$, so for two propositions *p* and *q*:

$p \wedge q$ Conjunction between two propositions. Say "*p* and *q*".

Conjunctions play a similar role to the word "and" in language and the set operation **intersection** (see study guide: *Operations on Sets*). If the letter *p* represents the basic proposition "It is sunny" and the letter *q* represents the basic proposition "It is hot". You can join them together as "It is sunny **and** it is hot" to make a new proposition written symbolically as $p \wedge q$. The truth of this new proposition is determined by the truth of *p* and *q*. The most common way of representing how conjunction (and other connectives) works is through a **truth table**. Truth tables list all the possible combinations of true and false (usually abbreviated to T and F) for the basic propositions and then give the corresponding true/false output for the connective. The truth table for conjunction is:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Which shows that conjunction is only true if both *p* and *q* are true, otherwise it is false. In other words $p \wedge q$ is only true if it is both sunny *and* it is hot. It is false, for example, if it is sunny *and* it is not hot.

2. Disjunction

Another way of joining two propositions together is called **disjunction** which is given the symbol $\vee$, so for two propositions *p* or *q*:

$p \vee q$ — Disjunction between two propositions. Say "*p* or *q*".

Disjunction play a similar role to the word "or" in language and the set operation **union** (see study guide: *Operations on Sets*). Again let *p* represent the basic proposition "It is sunny" and *q* represent the basic proposition "It is hot". You can join them together as "It is sunny **or** it is hot" to make a new proposition written symbolically as $p \vee q$. The truth table for disjunction is:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Which says that disjunction is true if either *p* or *q* (or both) are true. It is false only if they are both false. So for example $p \vee q$ is true if it is sunny *or* it is hot. It is false if it is neither sunny nor hot.

3. Negation

The negation of a proposition swaps true and false so a true proposition becomes false and a false proposition becomes true. There are two common symbols to denote negation of a proposition *p* by either $\overline{p}$ or $\neg p$. This guide will use the first so:

$\overline{p}$ — Negation of *p*: say "not *p*".

Negation plays a similar role to the word "not" in language and the set operation **complement** (see study guide: *Operations on Sets*). So if *p* is the proposition "It is sunny", the negation would be "It is not sunny". In other words $\overline{p}$ is true if it is not sunny. The truth table for negation is:

p	$\overline{p}$
T	F
F	T

4. Implication: the conditional connective

The conditional connective is used to make propositions of the form "if *this* then *that*". Commonly a conditional proposition is written as $p \rightarrow q$ (you may also see $p \Rightarrow q$ or $p \supset q$) and you say that "*p* implies *q*".

$p \rightarrow q$ — Conditional proposition. Say "if *p* then *q*" or "*p* implies *q*".

The truth table for the conditional connective is:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Which shows that a conditional proposition is only false when *p* is true and *q* is false. The last line of this table can be confusing. If $p \to q$ is taken to mean "*p* implies *q"* and "implies" is taken to mean "proves" or "causes" then if *p* is false it does not seem to imply anything and so you might think that $p \to q$ is false. However in logic it is better to take it to mean "if *p* then *q"* and, since *p* is false, it does not matter what *q* is.

5. If and only if: the biconditional connective

The biconditional proposition is used to make propositions of the form "*this* if and only if *that*". Commonly, the biconditional statement is written as $p \leftrightarrow q$ (you may also see $p \Leftrightarrow q$ or $p \equiv q$) and you say that "*p* if and only if *q*" which is often shortened to "iff".

$p \leftrightarrow q$ Biconditional proposition. Say "*p* if and only if *q*".

The truth table for the biconditional connective is:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Which shows that a biconditional proposition is true when *p* and *q* are either both true or both false, otherwise it is false.

Constructing truth tables: more complicated statements

You can use combinations of the connectives described above to make more complicated logical propositions and to test their truthfulness or falsehood for given inputs. You can do this by constructing truth tables which describe the propositions you are looking at. Let's look at the proposition $\overline{p} \vee q$. To construct the truth table you need to know how *p*, *q*, $\overline{p}$ and $\vee$ behave.

p	q
T	T
T	F
F	T
F	F

If you look at the tables in this guide they always begin by listing all the combinations of true and false that the propositions can take. You should take this approach when constructing the truth table so begin with the combinations of *p* and *q*.

p	q	$\overline{p}$
T	T	F
T	F	F
F	T	T
F	F	T

Next you need to consider $\overline{p}$, which is part of the proposition. You can negate the *p* column and put the result in a new column.

p	q	$\overline{p}$	$\overline{p} \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Finally you can use the $\overline{p}$ and *q* columns as the inputs to $\vee$ (remember that $\vee$ is only false when both inputs are false). This gives the required truth table.

This result is important as you may notice that the final column is the same as that for $p \rightarrow q$. In logic if two propositions have the same truth table then are said to be **logically equivalent**. Logical equivalence is denoted by $\equiv$ and so you write:

$$p \rightarrow q \equiv \overline{p} \vee q$$

"*p* implies *q*" is **logically equivalent** to "not *p* or *q*".

Tautologies and Contradictions

Sometimes a proposition is true regardless of the combination of inputs, these proposition are called **tautologies**. You can use truth tables to test whether propositions are tautologies. (Note: all tautologies are logically equivalent to each other.)

A **tautology** is a proposition which is true regardless of the true or false states of the basic propositions in it.

Example: Is $p \vee \overline{p}$ a tautology?

Construct the truth table using *p*, then $\overline{p}$ and finally $p \vee \overline{p}$, remembering that $\vee$ is only false when both inputs are false.

p	$\overline{p}$	$p \vee \overline{p}$
T	F	T
F	T	T

And so $p \vee \overline{p}$ is a tautology as it is always true regardless of whether p is true or false.

Sometimes a proposition is false regardless of the combination of inputs, these proposition are called **contradictions**. You can use truth tables to test whether propositions are contradictions, also all contradictions are logically equivalent to each other.

> A contradiction is a proposition which is false regardless of the true or false states of the propositions in it.

Example: Is $p \wedge \overline{p}$ a contradiction?

Construct the truth table using p, then $\overline{p}$ and finally $p \wedge \overline{p}$, remembering that $\wedge$ is only true when both inputs are true.

p	$\overline{p}$	$p \wedge \overline{p}$
T	F	F
F	T	F

Which shows that $p \wedge \overline{p}$ is a contradiction as the proposition is always false.

Want to know more?

If you have any further questions about this topic you can make an appointment to see a **Learning Enhancement Tutor** in the **Student Support Service**, as well as speaking to your lecturer or adviser.

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