Steps into Vectors

The Cross Product

This guide introduces the cross product of two vectors which is also known as the vector product. It describes how you can calculate it and explains some geometry associated with it.

Introduction

The study guide: *Basic Operations with Vectors* described how vectors can be added together, subtracted from each other or multiplied by a scalar. Multiplying vectors is not defined, however there are operations between two vectors that use the language and symbols of multiplication. Specifically these are the **dot product** (often called the **scalar product**) and the **cross product** (often called the **vector product**). You can see that word "product" has been used. You will also see that a dot or a cross is commonly used to denote these operations. This all implies multiplication of the vectors, but dot and cross products are not multiplication. This study guide is concerned with the cross product of two vectors which has many uses in geometry and physics such as playing a central role in Maxwell's Equations of electromagnetism, defining the sine rule, calculating areas of triangles and calculating forces. You can learn more about the dot product in the study guide: *The Dot Product*. You should read the study guide: *Basics of Vectors* before continuing with this guide if you are not familiar with calculating the **magnitude of a vector** or with the **rectangular unit vectors**.

How to calculate the cross product

The cross product can be defined one of two ways and it is common for it to be first introduced as follows.

For any two vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ the cross product of them is:

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}} \qquad \text{(say “a cross b”)}$$

Where θ is the angle between the two vectors.

$|\mathbf{a}|$ is the magnitude of **a**, often written as a: $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

$|\mathbf{b}|$ is the magnitude of **b**, often written as b: $|\mathbf{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$.

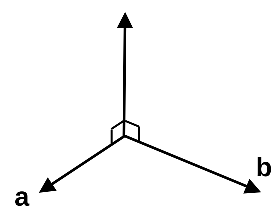
$\hat{\mathbf{n}}$ is the unit vector **normal** to both **a** and **b**.

It is common to see the symbol ^ used to denote the cross product but this guide will use the symbol $\times$.

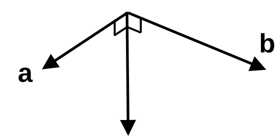
In the cross product formula $|\mathbf{a}|$, $|\mathbf{b}|$ and $\sin\theta$ are all scalars but $\hat{\mathbf{n}}$ is a vector and so the result of the cross product is a **vector** since you *can* multiply a vector by a scalar to obtain a vector. This is why the cross product is often referred to as the vector product.

The result of finding the cross product of two vectors is a vector.

Importantly the result of the cross product is **normal** to both **a** and **b**. This means the cross product gives a vector at right-angles (often said to be **orthogonal**) to both **a** and **b**. Two vectors with different directions are always coplanar (in the same plane) and two vectors with the same direction are collinear (in the same line). The diagram below shows the coplanar vectors **a** and **b** which are not necessarily orthogonal to each other. As you can see, you can have two possible vectors which are normal to them. So which one do you choose?



Normal pointing up

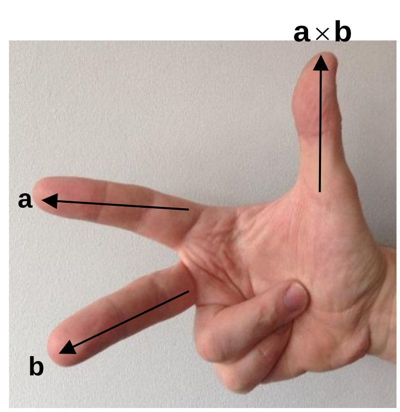


Normal pointing down

The result of a cross product follows the **right-hand rule**. In the diagram above the right-hand rule gives the result of $\mathbf{a}\times\mathbf{b}$ as the one on the left, with the normal pointing up.

To use the right-hand rule you:

1. Point the index finger of your right hand in the direction of **a**.
2. Point the middle finger of your right hand in the direction of **b**.
3. The thumb of your right hand will point in the direction of $\mathbf{a}\times\mathbf{b}$.

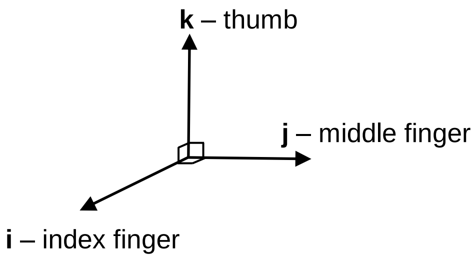


Finding the cross products of rectangular unit vectors

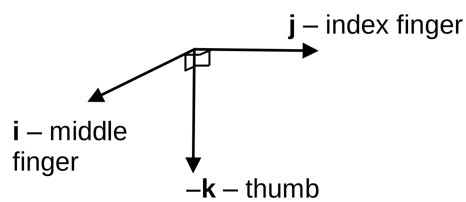
You can use the right-hand rule, in conjunction with the cross product formula on the first page of this guide, to find the cross products of the various rectangular unit vectors **i**, **j** and **k** with each other.

Example: What are $\mathbf{i} \times \mathbf{j}$ and $\mathbf{j} \times \mathbf{i}$?

Looking at the cross product formula $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \, \hat{\mathbf{n}}$, for $\mathbf{i} \times \mathbf{j}$ and $\mathbf{j} \times \mathbf{i}$ the magnitudes of the vectors are 1 and the sine of angle between them is $\sin 90^\circ = 1$. So $\mathbf{i} \times \mathbf{j}$ and $\mathbf{j} \times \mathbf{i}$ are equal to either **k** or $-\mathbf{k}$ as they are both normal to each of them. You can use the right-hand rule to determine which one is the answer you require:



$\mathbf{i} \times \mathbf{j} = \mathbf{k}$



$\mathbf{j} \times \mathbf{i} = -\mathbf{k}$

In fact, in general for any two vectors **a** and **b**:

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

Mathematically this means that the cross product does not **commute**. In other words it is important which way around **a** and **b** are written.

You can calculate the cross product of each pair of the rectangular unit vectors in this way and you find that:

$$\mathbf{i} \times \mathbf{k} = -\mathbf{j} \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j} \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i}$$

Example: What is $\mathbf{i} \times \mathbf{i}$?

Remember that the magnitude of **i** (and also **j** and **k**) is 1. Also the angle between **i** and **i** is 0° as the vectors have the same direction so:

$$\mathbf{i} \times \mathbf{i} = |\mathbf{i}||\mathbf{i}| \sin 0^\circ \, \hat{\mathbf{n}} = \mathbf{0}$$

as $\sin 0° = 0$. This is also the case for $\mathbf{j} \times \mathbf{j}$ and $\mathbf{k} \times \mathbf{k}$ as the angle between the vectors is zero in both cases. Remember that **0** is called the **null vector**.

The cross product of two vectors with the same direction is the null vector.

The results from these two examples are extremely useful for finding an alternative way of calculating the cross product as you shall see in the next section.

An alternative formula for the cross product

You now have enough information to work out an alternative way of calculating the cross product of two vectors. You can think of $\mathbf{a} \times \mathbf{b}$ as a process similar to opening the brackets. In this respect the cross product behaves like multiplication (technically it is called **distributivity**) but the actual calculation of the cross product is not multiplication as mentioned above. You can write:

$$\mathbf{a} \times \mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$$

Here the coefficients of the rectangular unit vectors are simply multiplied together as they are just numbers and you take the relevant cross products of the rectangular unit vectors remembering to take extra care over their order. After opening the brackets you get:

$$\begin{aligned}\mathbf{a} \times \mathbf{b} = {} & a_1b_1\,\mathbf{i} \times \mathbf{i} + a_1b_2\,\mathbf{i} \times \mathbf{j} + a_1b_3\,\mathbf{i} \times \mathbf{k} \\ & + a_2b_1\,\mathbf{j} \times \mathbf{i} + a_2b_2\,\mathbf{j} \times \mathbf{j} + a_2b_3\,\mathbf{j} \times \mathbf{k} \\ & + a_3b_1\,\mathbf{k} \times \mathbf{i} + a_3b_2\,\mathbf{k} \times \mathbf{j} + a_3b_3\,\mathbf{k} \times \mathbf{k}\end{aligned}$$

Using the results of the rectangular unit vectors in the examples in the previous section, you can write the result in terms of the coefficients of **i**, **j** and **k**:

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

This formula is an important, alternative way of calculating the cross product of two vectors.

It is often expressed as a **determinant**:

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

whose calculation gives exactly the same formula. A determinant is a piece of mathematics from the study of matrices. If you are not familiar with matrices then it is you can read the study guide: *Calculating Determinants* to help you understand what this means.

The formula above offers a much easier way of calculating the cross product of two vectors as you do not need to know the angle between them or the normal vector to find the result.

Example: Find a vector that is normal to both the vectors $\mathbf{a}=3\mathbf{i}+2\mathbf{j}+4\mathbf{k}$ and $\mathbf{b}=\mathbf{i}-2\mathbf{j}+3\mathbf{k}$.

You can use the cross product to find the vector that is normal to $\mathbf{a}=3\mathbf{i}+2\mathbf{j}+4\mathbf{k}$ and $\mathbf{b}=\mathbf{i}-2\mathbf{j}+3\mathbf{k}$. As $a_1=3$, $a_2=2$, $a_3=4$, $b_1=1$, $b_2=-2$ and $b_3=3$:

$$\begin{aligned}\mathbf{a}\times\mathbf{b}&=(a_2b_3-a_3b_2)\mathbf{i}-(a_1b_3-a_3b_1)\mathbf{j}+(a_1b_2-a_2b_1)\mathbf{k}\\&=(6-(-8))\mathbf{i}-(9-4)\mathbf{j}+(-6-2)\mathbf{k}\\&=14\mathbf{i}-5\mathbf{j}-8\mathbf{k}\end{aligned}$$

Which tells you that the vector $14\mathbf{i}-5\mathbf{j}-8\mathbf{k}$ is normal to both $\mathbf{a}=3\mathbf{i}+2\mathbf{j}+4\mathbf{k}$ and $\mathbf{b}=\mathbf{i}-2\mathbf{j}+3\mathbf{k}$.

You can test whether two vectors are orthogonal by using the dot product as the dot product of two orthogonal vectors is zero, see study guide: *The Dot Product*. In other words checking that the dot products of the result with both **a** and **b** are zero.

For **a**:

$$(3\mathbf{i}+2\mathbf{j}+4\mathbf{k})\cdot(14\mathbf{i}-5\mathbf{j}-8\mathbf{k})=3\cdot14+2\cdot(-5)+4\cdot(-8)=42-10-32=0$$

Also for **b**:

$$(\mathbf{i}-2\mathbf{j}+3\mathbf{k})\cdot(14\mathbf{i}-5\mathbf{j}-8\mathbf{k})=1\cdot14+(-2)\cdot(-5)+3\cdot(-8)=14+10-24=0$$

Example: What is the cross product of the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$ and what does it tell you?

Here $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $b_1 = 2$, $b_2 = 4$ and $b_3 = 6$ so:

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k} \\ &= (12 - 12)\mathbf{i} - (6 - 6)\mathbf{j} + (4 - 4)\mathbf{k} \\ &= \mathbf{0}\end{aligned}$$

Earlier in this guide you found that vectors that have the same direction have a cross product equal to the null vector. In this example $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ which implies that **a** and **b** must have the same direction. If you look carefully at **a** and **b** you can see that $\mathbf{b} = 2\mathbf{a}$ and so the vectors do indeed have the same direction as one is a multiple of the other.

Want to know more?

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