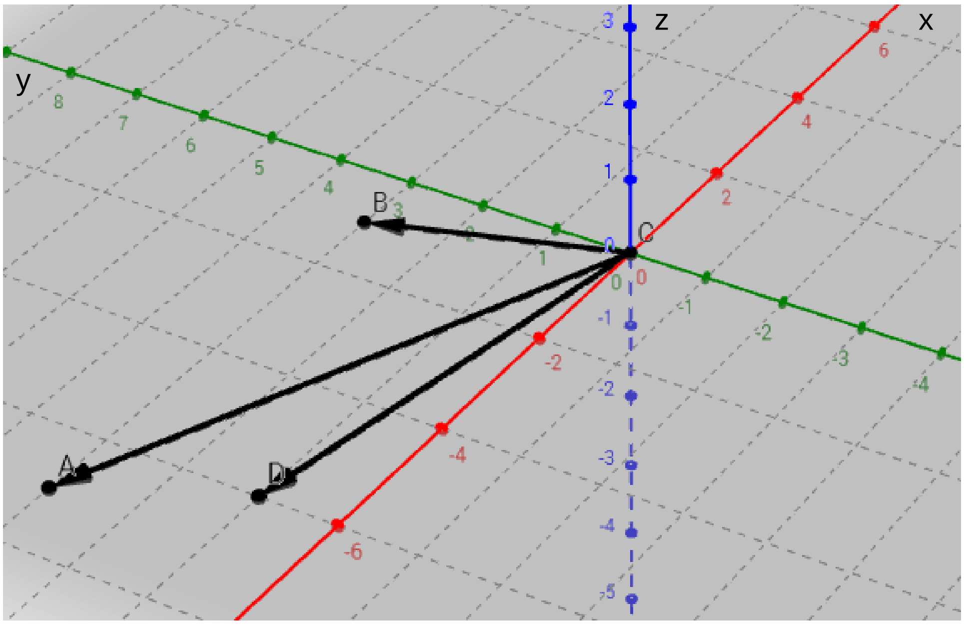
Learning Enhancement Team

Worksheet: The Cross Product

These are the model answers for the worksheet that has questions on the cross product between vectors.

The Cross Product study guide

1.



a. Looking at image, you can see that the vectors $\overrightarrow{CD}$, $\overrightarrow{CA}$ and $\overrightarrow{CB}$ in terms of the rectangular unit vectors **i**, **j** and **k** are:

$$\overrightarrow{CD} = -6\mathbf{i} + 1\mathbf{j} + 0\mathbf{k},$$

$$\overrightarrow{CA} = -7\mathbf{i} + 3\mathbf{j} + 0\mathbf{k} \text{ and}$$

$$\overrightarrow{CB} = -1\mathbf{i} + 3\mathbf{j} + 0\mathbf{k}.$$

b. The cross product of two vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ in **3 Dimensional space** (3D) is given by:

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

i. For $\overrightarrow{CD} = \mathbf{a}$ and $\overrightarrow{CA} = \mathbf{b}$ you have that $a_1 = -6$, $a_2 = 1$, $a_3 = 0$, $b_1 = -7$, $b_2 = 3$ and $b_3 = 0$:

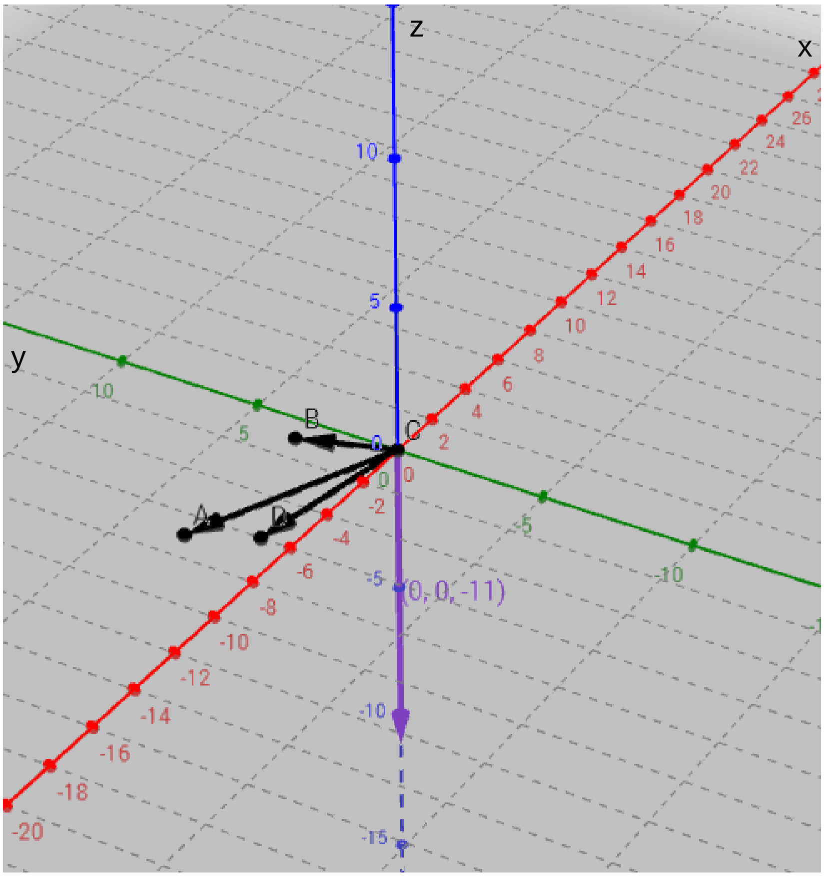
$$\overrightarrow{CD} \times \overrightarrow{CA} = \mathbf{a} \times \mathbf{b}$$

$$= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

$$= (1 \cdot 0 - 0 \cdot 3)\mathbf{i} - (-6 \cdot 0 - 0 \cdot (-7))\mathbf{j} + (-6 \cdot 3 - 1 \cdot (-7))\mathbf{k}$$

$$= 0\mathbf{i} + 0\mathbf{j} - 11\mathbf{k}$$

In the image below, you can see the vector $\overrightarrow{CD} \times \overrightarrow{CA}$ coloured purple. The result of the cross product is a vector **orthogonal** (at a right-angle) to both $\overrightarrow{CD}$ and $\overrightarrow{CA}$. You can also observe that the x and y-coordinates of this vector are zero.



ii. For $\overrightarrow{CA} = \mathbf{a}$ and $\overrightarrow{CD} = \mathbf{b}$ you have that $a_1 = -7$, $a_2 = 3$, $a_3 = 0$, $b_1 = -6$, $b_2 = 1$ and $b_3 = 0$:

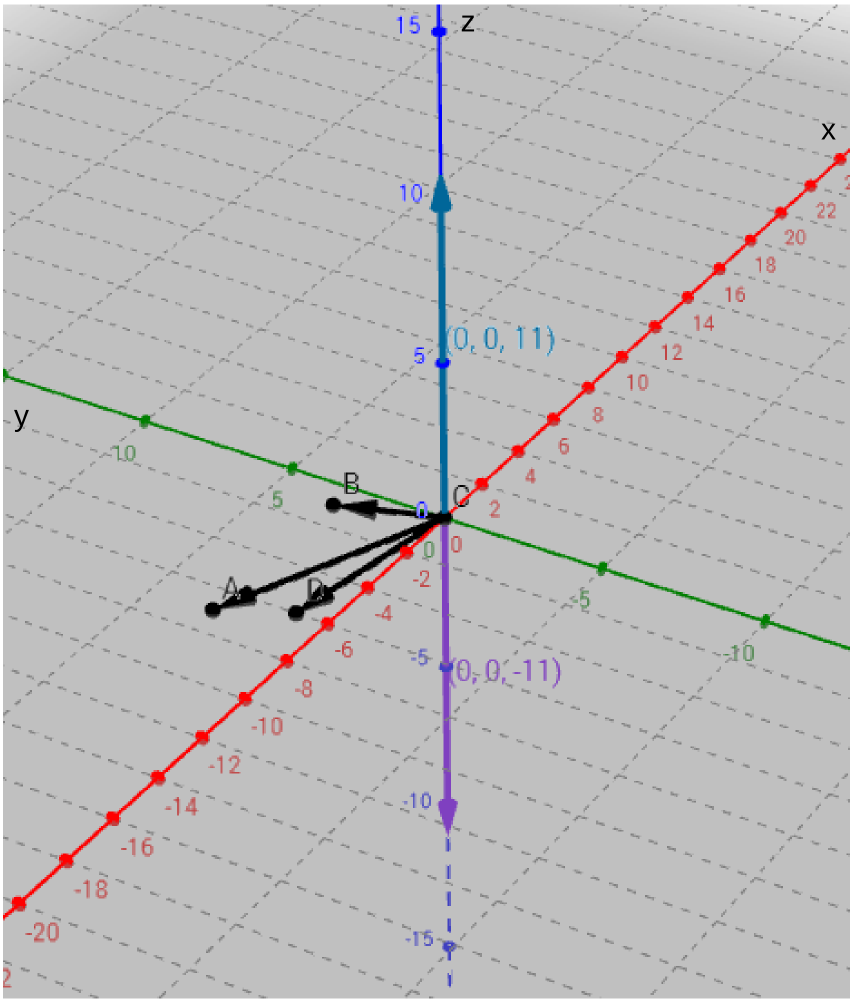
$$\overrightarrow{CA} \times \overrightarrow{CD} = \mathbf{a} \times \mathbf{b}$$

$$= (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$$

$$= (3 \cdot 0 - 0 \cdot 1)\mathbf{i} - (-7 \cdot 0 - 0 \cdot (-6))\mathbf{j} + (-7 \cdot 1 - 3 \cdot (-6))\mathbf{k}$$

$$= 0\mathbf{i} + 0\mathbf{j} + 11\mathbf{k}$$

In the image below, you can see the vector $\overrightarrow{CA} \times \overrightarrow{CD}$ coloured petrol blue. The result of the cross product is a vector **orthogonal** (at a right-angle) to both $\overrightarrow{CA}$ and $\overrightarrow{CD}$. You can also observe that the x and y-coordinates of this vector are zero.



iii. For $\overrightarrow{CD} = \mathbf{a}$ and $\overrightarrow{CB} = \mathbf{b}$ you have that $a_1 = -6$, $a_2 = 1$, $a_3 = 0$, $b_1 = -1$, $b_2 = 3$ and $b_3 = 0$:

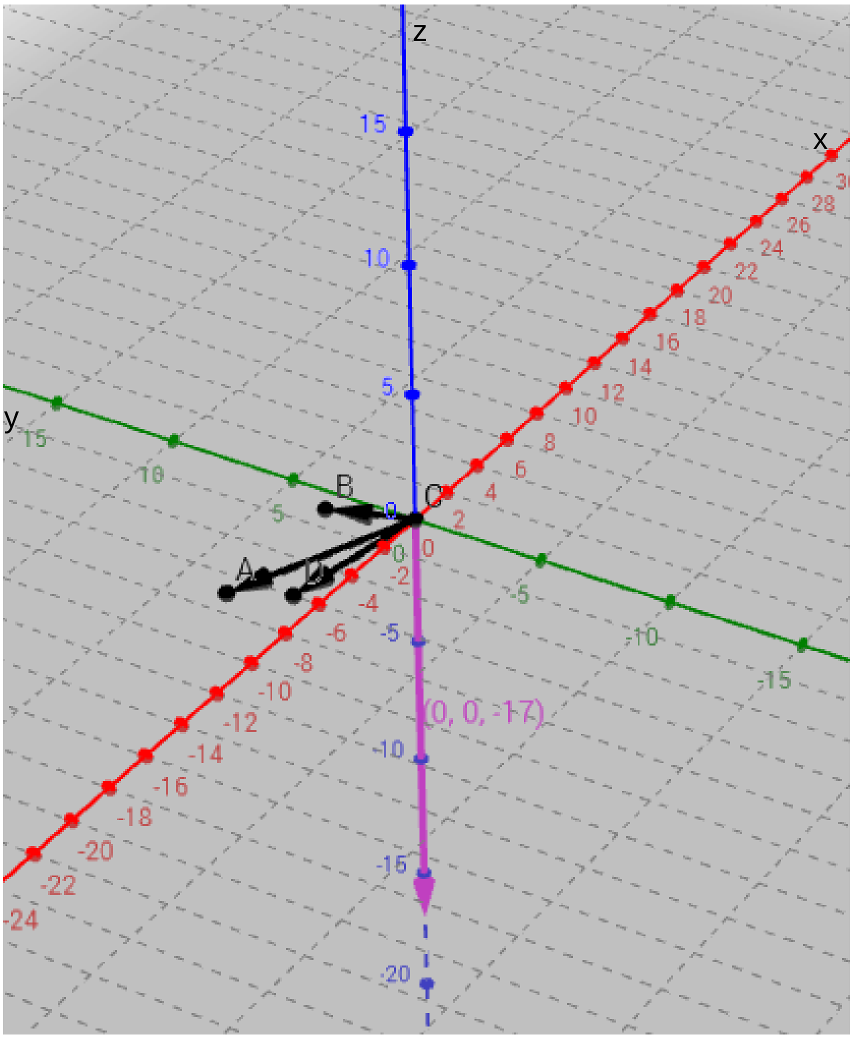
$$\overrightarrow{CD} \times \overrightarrow{CB} = \mathbf{a} \times \mathbf{b}$$

$$= (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$$

$$= (1 \cdot 0 - 0 \cdot 3)\mathbf{i} - (-6 \cdot 0 - 0 \cdot (-1))\mathbf{j} + (-6 \cdot 3 - 1 \cdot (-1))\mathbf{k}$$

$$= 0\mathbf{i} + 0\mathbf{j} - 17\mathbf{k}$$

In the image below, you can see the vector $\overrightarrow{CD} \times \overrightarrow{CB}$ coloured pink. The result of the cross product is a vector **orthogonal** (at a right-angle) to both $\overrightarrow{CD}$ and $\overrightarrow{CB}$. You can also observe that the x and y-coordinates of this vector are zero



iv. For $\overrightarrow{CB} = \mathbf{a}$ and $\overrightarrow{CD} = \mathbf{b}$ you have that $a_1 = -1$, $a_2 = 3$, $a_3 = 0$, $b_1 = -6$, $b_2 = 1$ and $b_3 = 0$:

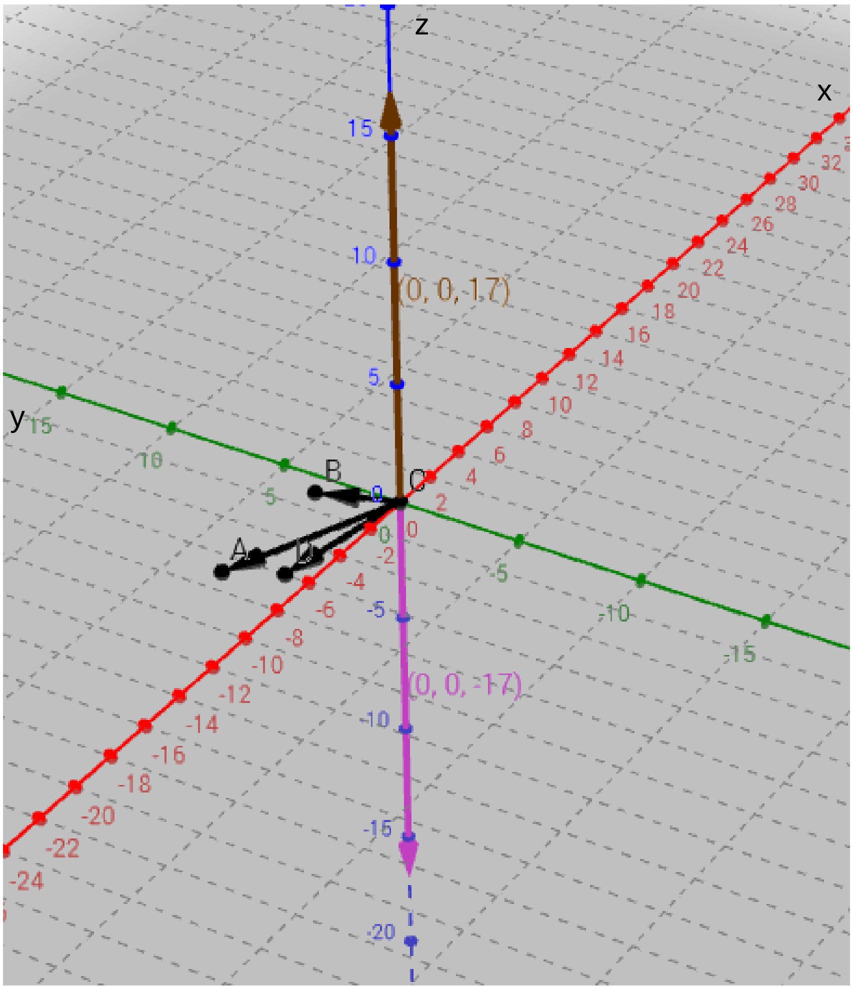
$$\overrightarrow{CB} \times \overrightarrow{CD} = \mathbf{a} \times \mathbf{b}$$

$$= (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$$

$$= (3 \cdot 0 - 0 \cdot 1)\mathbf{i} - (-1 \cdot 0 - 0 \cdot (-6))\mathbf{j} + (-1 \cdot 1 - 3 \cdot (-6))\mathbf{k}$$

$$= 0\mathbf{i} + 0\mathbf{j} + 17\mathbf{k}$$

In the image below, you can see the vector $\overrightarrow{CB} \times \overrightarrow{CD}$ coloured brown. The result of the cross product is a vectors **orthogonal** (at a right-angle) to both $\overrightarrow{CB}$ and $\overrightarrow{CD}$. You can also observe that the x and y-coordinates of this vector are zero.



c. Looking at your answers to parts (i) and (ii) you notice that:

$$\overrightarrow{CD}\times\overrightarrow{CA}=0\mathbf{i}+0\,\mathbf{j}-11\mathbf{k}=-\left(0\mathbf{i}+0\,\mathbf{j}+11\mathbf{k}\right)=-\left(\overrightarrow{CA}\times\overrightarrow{CD}\right)$$

And looking at your answers to parts (iii) and (iv) you can see that:

$$\overrightarrow{CD}\times\overrightarrow{CB}=0\mathbf{i}+0\,\mathbf{j}-17\mathbf{k}=-\left(0\mathbf{i}+0\,\mathbf{j}+17\mathbf{k}\right)=-\left(\overrightarrow{CB}\times\overrightarrow{CD}\right)$$

So a piece of mathematics which describes this is:

"For any vectors **a** and **b**: $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$"

2. The cross product of two vectors $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ in **3 Dimensional space** (3D) is given by:

$$\mathbf{a}\times\mathbf{b}=\left(a_2b_3-a_3b_2\right)\mathbf{i}-\left(a_1b_3-a_3b_1\right)\mathbf{j}+\left(a_1b_2-a_2b_1\right)\mathbf{k}$$

a. For $\mathbf{a}=2\mathbf{i}+3\mathbf{j}-4\mathbf{k}$ and $\mathbf{b}=-2\mathbf{i}+\mathbf{j}-2\mathbf{k}$ you have that $a_1=2$, $a_2=3$, $a_3=-4$, $b_1=-2$, $b_2=1$ and $b_3=-2$. So,

$$\mathbf{a}\times\mathbf{b}=\left(a_2b_3-a_3b_2\right)\mathbf{i}-\left(a_1b_3-a_3b_1\right)\mathbf{j}+\left(a_1b_2-a_2b_1\right)\mathbf{k}$$

$$=\left(-6-(-4)\right)\mathbf{i}-\left(-4-8\right)\mathbf{j}+\left(2-(-6)\right)\mathbf{k}$$

$$=-2\,\mathbf{i}+12\,\mathbf{j}+8\mathbf{k}$$

Which tells you that the vector $-2\mathbf{i}+12\mathbf{j}+8\mathbf{k}$ is normal (orthogonal) to both $\mathbf{a}=2\mathbf{i}+3\mathbf{j}-4\mathbf{k}$ and $\mathbf{b}=-2\mathbf{i}+\mathbf{j}-2\mathbf{k}$.

You can test whether two vectors are orthogonal by using the dot product as the dot product of two orthogonal vectors is zero, see study guide: The Dot Product. In other words checking that the dot products of the result with both **a** and **b** are zero.

For **a**:

$$(2\mathbf{i}+3\mathbf{j}-4\mathbf{k})\cdot(-2\mathbf{i}+12\mathbf{j}+8\mathbf{k})=2\cdot(-2)+3\cdot 12+(-4)\cdot 8=-4+36-32=0$$

Also for **b**:

$$(-2\mathbf{i}+\mathbf{j}-2\mathbf{k})\cdot(-2\mathbf{i}+12\mathbf{j}+8\mathbf{k})=(-2)\cdot(-2)+1\cdot 12+(-2)\cdot 8=4+12-16=0$$

b. For $\mathbf{a}=\frac{2}{3}\mathbf{i}+\frac{1}{2}\mathbf{j}-\mathbf{k}$ and $\mathbf{b}=-\mathbf{i}+\frac{1}{3}\mathbf{j}-\frac{1}{2}\mathbf{k}$, you have that $a_1=\frac{2}{3}$, $a_2=\frac{1}{2}$, $a_3=-1$, $b_1=-1$, $b_2=\frac{1}{3}$ and $b_3=-\frac{1}{2}$. So,

$$\mathbf{a}\times\mathbf{b}=(a_2b_3-a_3b_2)\mathbf{i}-(a_1b_3-a_3b_1)\mathbf{j}+(a_1b_2-a_2b_1)\mathbf{k}$$

$$=\left(-\tfrac{1}{4}-\left(-\tfrac{1}{3}\right)\right)\mathbf{i}-\left(-\tfrac{1}{3}-1\right)\mathbf{j}+\left(\tfrac{2}{9}-\left(-\tfrac{1}{2}\right)\right)\mathbf{k}$$

$$=\tfrac{1}{12}\mathbf{i}+\tfrac{4}{3}\mathbf{j}+\tfrac{13}{18}\mathbf{k}$$

Which tells you that the vector $\frac{1}{12}\mathbf{i}+\frac{4}{3}\mathbf{j}+\frac{13}{18}\mathbf{k}$ is normal (orthogonal) to both $\mathbf{a}=\frac{2}{3}\mathbf{i}+\frac{1}{2}\mathbf{j}-\mathbf{k}$ and $\mathbf{b}=-\mathbf{i}+\frac{1}{3}\mathbf{j}-\frac{1}{2}\mathbf{k}$.

You can test whether two vectors are orthogonal by using the dot product as the dot product of two orthogonal vectors is zero, see study guide: The Dot Product. In other words checking that the dot products of the result with both **a** and **b** are zero.

For **a**:

$$\left(\tfrac{2}{3}\mathbf{i}+\tfrac{1}{2}\mathbf{j}-\mathbf{k}\right)\cdot\left(\tfrac{1}{12}\mathbf{i}+\tfrac{4}{3}\mathbf{j}+\tfrac{13}{18}\mathbf{k}\right)=\tfrac{2}{3}\cdot\tfrac{1}{12}+\tfrac{1}{2}\cdot\tfrac{4}{3}+(-1)\cdot\tfrac{13}{18}=\tfrac{1}{18}+\tfrac{12}{18}-\tfrac{13}{18}=0$$

Also for **b**:

$$\left(-\mathbf{i}+\tfrac{1}{3}\mathbf{j}-\tfrac{1}{2}\mathbf{k}\right)\cdot\left(\tfrac{1}{12}\mathbf{i}+\tfrac{4}{3}\mathbf{j}+\tfrac{13}{18}\mathbf{k}\right)=(-1)\cdot\tfrac{1}{12}+\tfrac{1}{3}\cdot\tfrac{4}{3}+\left(-\tfrac{1}{2}\right)\cdot\tfrac{13}{18}=-\tfrac{3}{36}+\tfrac{16}{36}-\tfrac{13}{36}=0$$

c. For $\mathbf{a}=-\mathbf{i}+\sqrt{2}\mathbf{j}-\mathbf{k}$ and $\mathbf{b}=\sqrt{2}\mathbf{i}+3\mathbf{j}+4\mathbf{k}$, you have that $a_1=-1$, $a_2=\sqrt{2}$, $a_3=-1$, $b_1=\sqrt{2}$, $b_2=3$ and $b_3=4$. So,

$$\mathbf{a}\times\mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

$$= \left(4\sqrt{2} - (-3)\right)\mathbf{i} - \left(-4 - \left(-\sqrt{2}\right)\right)\mathbf{j} + (-3 - 2)\mathbf{k}$$

$$= \left(3 + 4\sqrt{2}\right)\mathbf{i} + \left(4 - \sqrt{2}\right)\mathbf{j} - 5\mathbf{k}$$

Which tells you that the vector $\left(3+4\sqrt{2}\right)\mathbf{i} + \left(4-\sqrt{2}\right)\mathbf{j} - 5\mathbf{k}$ is normal (orthogonal) to both $\mathbf{a} = -\mathbf{i} + \sqrt{2}\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \sqrt{2}\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$.

You can test whether two vectors are orthogonal by using the dot product as the dot product of two orthogonal vectors is zero, see study guide: The Dot Product. In other words checking that the dot products of the result with both **a** and **b** are zero.

For **a**:

$$\left(-\mathbf{i} + \sqrt{2}\mathbf{j} - \mathbf{k}\right)\cdot\left(\left(3 + 4\sqrt{2}\right)\mathbf{i} + \left(4 - \sqrt{2}\right)\mathbf{j} - 5\mathbf{k}\right) = -1\cdot\left(3 + 4\sqrt{2}\right) + \sqrt{2}\cdot\left(4 - \sqrt{2}\right) + (-1)\cdot(-5)$$

$$= -3 - 4\sqrt{2} + 4\sqrt{2} - 2 + 5$$

$$= 0$$

Also for **b**:

$$\left(\sqrt{2}\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}\right)\cdot\left(\left(3 + 4\sqrt{2}\right)\mathbf{i} + \left(4 - \sqrt{2}\right)\mathbf{j} - 5\mathbf{k}\right) = \sqrt{2}\cdot\left(3 + 4\sqrt{2}\right) + 3\cdot\left(4 - \sqrt{2}\right) + 4\cdot(-5)$$

$$= 3\sqrt{2} + 8 + 12 - 3\sqrt{2} - 20$$

$$= 0$$

3. You are given the following vectors $\mathbf{a} = -2\mathbf{i} - 5\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} - 1\mathbf{j} + 3\mathbf{k}$.

a. In order to find a vector **c** that is normal (orthogonal) to both vectors **a** and **b**, you need to calculate their cross product: $\mathbf{a}\times\mathbf{b}$.

The cross product of two vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ in **3 Dimensional space** (3D) is given by:

$$\mathbf{a}\times\mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

You can use the cross product to find the vector that is normal (orthogonal) to $\mathbf{a} = -2\mathbf{i} - 5\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} - 1\mathbf{j} + 3\mathbf{k}$. As $a_1 = -2$, $a_2 = -5$, $a_3 = 1$, $b_1 = 4$, $b_2 = -1$ and $b_3 = 3$:

$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$$

$$= (-15 - (-1))\mathbf{i} - (-6 - 4)\mathbf{j} + (2 - (-20))\mathbf{k}$$

$$= -14\mathbf{i} + 10\mathbf{j} + 22\mathbf{k}$$

So, $\mathbf{c} = -14\mathbf{i} + 10\mathbf{j} + 22\mathbf{k}$

b. In order to verify that your vector **c** is normal (orthogonal) to both **a** and **b**, you need to check that the dot products $\mathbf{a} \cdot \mathbf{c}$ and $\mathbf{b} \cdot \mathbf{c}$ are zero.

For **a**:

$$(-2\mathbf{i} - 5\mathbf{j} + \mathbf{k}) \cdot (-14\mathbf{i} + 10\mathbf{j} + 22\mathbf{k}) = -2 \cdot (-14) + (-5) \cdot 10 + 1 \cdot 22$$

$$= +28 - 50 + 22 = 0$$

Also for **b**:

$$(4\mathbf{i} - 1\mathbf{j} + 3\mathbf{k}) \cdot (-14\mathbf{i} + 10\mathbf{j} + 22\mathbf{k}) = 4 \cdot (-14) - 1 \cdot 10 + 3 \cdot 22$$

$$= -56 - 10 + 66 = 0$$

c. You can calculate the magnitude of **c** using the following:

$$|\mathbf{c}| = \sqrt{c_1^2 + c_2^2 + c_3^2} = \sqrt{(-14)^2 + (10)^2 + (22)^2} = \sqrt{780} = 27.93 \text{ in 2 d.p.}$$

d. The cross product of two vectors **a** and **b** in the **3 Dimensional space** (3D) is also given by the following formula: $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \, \hat{\mathbf{n}}$. The magnitude of the cross product is given by: $|\mathbf{a} \times \mathbf{b}| = \left||\mathbf{a}||\mathbf{b}| \sin\theta \, \hat{\mathbf{n}}\right| = |\mathbf{a}||\mathbf{b}| \sin\theta |\hat{\mathbf{n}}|$. The vector $\hat{\mathbf{n}}$ is the unit vector **normal** to both **a** and **b,** so $|\hat{\mathbf{n}}| = 1$. And so:

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta |\hat{\mathbf{n}}| = |\mathbf{a}||\mathbf{b}| \sin\theta$$

To find the angle θ between the vectors **a** and **b** you need to solve $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$ for θ. Rearranging the equation $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$ for $\sin\theta$ you have $\sin\theta = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$.

You now need to find $\frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$. From part c of the question, you have that

$|\mathbf{a} \times \mathbf{b}| = |\mathbf{c}| = \sqrt{780} = 27.93$ in 2 d.p. You need to find the magnitude of the vectors **a** and **b**.

You can calculate the magnitude of **a** using the following:

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{(-2)^2 + (-5)^2 + (1)^2} = \sqrt{30}$$

And the magnitude of **b**

$$|\mathbf{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2} = \sqrt{(4)^2 + (-1)^2 + (3)^2} = \sqrt{26}$$

So, $\sin\theta = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}||\mathbf{b}|} = \frac{\sqrt{780}}{\sqrt{30} \cdot \sqrt{26}} = 1$. Solving the trigonometric equation you find that the angle between the two vectors is $\theta = 90°$. You can check your answer using the dot product.

4.

a. The two vectors **a** and **b** are in the **3 Dimensional space** (3D). So, their cross product is given by:

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

As $a_1 = \frac{1}{4}$, $a_2 = 2$, $a_3 = -1$, $b_1 = 1$, $b_2 = 8$ and $b_3 = -4$, their cross product is:

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

$$= (-8 - (-8))\mathbf{i} - (-1 - (-1))\mathbf{j} + (2 - 2)\mathbf{k}$$

$$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$$

The cross product of the two vectors is the null vector. This means that the vectors **a** and **b** have the same direction. You can also observe this using the following formula:

$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$$

The magnitude of the cross product is given by: $|\mathbf{a}\times\mathbf{b}| = \left||\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}\right| = |\mathbf{a}||\mathbf{b}|\sin\theta|\hat{\mathbf{n}}|$.

The vector $\hat{\mathbf{n}}$ is the unit vector **normal** to both **a** and **b,** so $|\hat{\mathbf{n}}| = 1$. And so:

$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta|\hat{\mathbf{n}}| = |\mathbf{a}||\mathbf{b}|\sin\theta$$

To find the angle θ between the vectors **a** and **b** you need to solve $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$ for θ. Rearranging the equation $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$ for $\sin\theta$ you have $\sin\theta = \dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$.

You have found that $\mathbf{a}\times\mathbf{b} = 0\mathbf{i}+0\mathbf{j}+0\mathbf{k}$. So, $|\mathbf{a}\times\mathbf{b}| = 0$ and $\sin\theta = \dfrac{0}{|\mathbf{a}||\mathbf{b}|} = 0$.
Solving the trigonometric equation you find that the angle between the two vectors is either $\theta = 0°$ or $\theta = 180°$. Checking the vectors **a** and **b** you can see whether they have the same (angle $\theta = 0°$) or opposite (angle $\theta = 180°$) direction and decide which of the angles is the right one. You can see that $\mathbf{a} = \frac{1}{4}\mathbf{b}$, so the vectors have the same direction and $\theta = 0°$.

b. To find the angle φ between the two vectors **a** and **c**, you need to find the cross product and then using the formula $|\mathbf{a}\times\mathbf{c}| = |\mathbf{a}||\mathbf{c}|\sin\phi$ find the angle between them.

The two vectors **a** and **c** are in the **3 Dimensional space** (3D). So, their cross product is given by:

$$\mathbf{a}\times\mathbf{c} = (a_2c_3 - a_3c_2)\mathbf{i} - (a_1c_3 - a_3c_1)\mathbf{j} + (a_1c_2 - a_2c_1)\mathbf{k}$$

As $a_1 = \frac{1}{4}$, $a_2 = 2$, $a_3 = -1$, $c_1 = -1$, $c_2 = -8$ and $c_3 = 4$, their cross product is:

$$\mathbf{a}\times\mathbf{c} = (a_2c_3 - a_3c_2)\mathbf{i} - (a_1c_3 - a_3c_1)\mathbf{j} + (a_1c_2 - a_2c_1)\mathbf{k}$$

$$= (8-8)\mathbf{i} - (1-1)\mathbf{j} + (-2-(-2))\mathbf{k}$$

$$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$$

The cross product of the two vectors is the null vector. This means that the vectors **a** and **c** have the same or opposite direction. You can also observe this using the following formula:

$$\mathbf{a}\times\mathbf{c} = |\mathbf{a}||\mathbf{c}|\sin\phi\hat{\mathbf{n}}$$

The magnitude of the cross product is given by: $|\mathbf{a}\times\mathbf{c}| = \left||\mathbf{a}||\mathbf{c}|\sin\varphi\hat{\mathbf{n}}\right| = |\mathbf{a}||\mathbf{c}|\sin\varphi|\hat{\mathbf{n}}|$.

The vector $\hat{\mathbf{n}}$ is the unit vector **normal** to both **a** and **c**, so $|\hat{\mathbf{n}}| = 1$. And so:

$$|\mathbf{a}\times\mathbf{c}| = |\mathbf{a}||\mathbf{c}|\sin\varphi|\hat{\mathbf{n}}| = |\mathbf{a}||\mathbf{c}|\sin\varphi$$

To find the angle ϕ between the vectors **a** and **c** you need to solve $|\mathbf{a}\times\mathbf{c}| = |\mathbf{a}||\mathbf{c}|\sin\varphi$ for ϕ. Rearranging the equation $|\mathbf{a}\times\mathbf{c}| = |\mathbf{a}||\mathbf{c}|\sin\varphi$ for $\sin\phi$ you have $\sin\phi = \dfrac{|\mathbf{a}\times\mathbf{c}|}{|\mathbf{a}||\mathbf{c}|}$.

You have found that $\mathbf{a}\times\mathbf{c} = 0\mathbf{i}+0\mathbf{j}+0\mathbf{k}$. So, $|\mathbf{a}\times\mathbf{c}| = 0$ and $\sin\phi = \dfrac{0}{|\mathbf{a}||\mathbf{c}|} = 0$.

Solving the trigonometric equation you find that the angle between the two vectors is either $\phi = 0°$ or $\phi = 180°$. Checking the vectors **a** and **c** you can see whether they have the same (angle $\phi = 0°$) or opposite (angle $\phi = 180°$) direction and decide which of the angles is the right one. You can see that $\mathbf{a} = -\frac{1}{4}\mathbf{c}$, so the vectors have the opposite direction and $\phi = 180°$.

5.

a. First you need to find the cross product $\mathbf{a}\times\mathbf{b}$

As $a_1 = -1$, $a_2 = -2$, $a_3 = 2$, $b_1 = 3$, $b_2 = -1$ and $b_3 = 1$:

Their cross product is:

$$\mathbf{a}\times\mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

$$= (-2-(-2))\mathbf{i} - (-1-6)\mathbf{j} + (1-(-6))\mathbf{k}$$

$$= 0\mathbf{i} + 7\mathbf{j} + 7\mathbf{k}$$

Then calculating the cross product $\mathbf{b}\times\mathbf{a}$ you find that:

$$\mathbf{b}\times\mathbf{a} = (b_2a_3 - b_3a_2)\mathbf{i} - (b_1a_3 - b_3a_1)\mathbf{j} + (b_1a_2 - b_2a_1)\mathbf{k}$$

$$= (-2-(-2))\mathbf{i} - (6-(-1))\mathbf{j} + (-6-1)\mathbf{k}$$

$$= 0\mathbf{i} - 7\mathbf{j} - 7\mathbf{k}$$

This confirms what you observed in question 1c. That:

$$\mathbf{a}\times\mathbf{b} = 0\mathbf{i} + 7\mathbf{j} + 7\mathbf{k} = -(0\mathbf{i} - 7\mathbf{j} - 7\mathbf{k}) = -\mathbf{b}\times\mathbf{a}$$

So $\mathbf{a}\times\mathbf{b}$ is in the opposite direction of $\mathbf{b}\times\mathbf{a}$

b. To calculate $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ with $\mathbf{c} = 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, you first need to calculate the cross product $\mathbf{b}\times\mathbf{c}$.

$$\mathbf{b}\times\mathbf{c} = (b_2c_3 - b_3c_2)\mathbf{i} - (b_1c_3 - b_3c_1)\mathbf{j} + (b_1c_2 - b_2c_1)\mathbf{k}$$

$$= (-1\cdot 3 - 1\cdot(-2))\mathbf{i} - (3\cdot 3 - 1\cdot 2)\mathbf{j} + (3\cdot(-2) - (-1)\cdot 2)\mathbf{k}$$

$$= -\mathbf{i} - 7\mathbf{j} - 4\mathbf{k}$$

And now to calculate the dot product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$: (to remind yourself how to calculate the dot product of two vectors see study guide: The Dot Product)

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (-\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})\cdot(-\mathbf{i} - 7\mathbf{j} - 4\mathbf{k})$$

$$= (-1)\cdot(-1) + (-2)\cdot(-7) + 2\cdot(-4)$$

$$= 1 + 14 - 8$$

$$= 7$$

To calculate $\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$, you first need to calculate the cross product $\mathbf{a}\times\mathbf{c}$.

$$\mathbf{c}\times\mathbf{a} = (c_2a_3 - c_3a_2)\mathbf{i} - (c_1a_3 - c_3a_1)\mathbf{j} + (c_1a_2 - c_2a_1)\mathbf{k}$$

$$= (-4-(-6))\mathbf{i} - (4-(-3))\mathbf{j} + (-4-2)\mathbf{k}$$

$$= 2\mathbf{i} - 7\mathbf{j} - 6\mathbf{k}$$

And now to calculate the dot product $\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$: (to remind yourself how to calculate the dot product of two vectors see study guide: The Dot Product)

$$\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=(3\mathbf{i}-\mathbf{j}+\mathbf{k})\cdot(2\mathbf{i}-7\mathbf{j}-6\mathbf{k})$$

$$=3\cdot 2+(-1)\cdot(-7)+1\cdot(-6)$$

$$=6+7-6$$

$$=7$$

Looking at your answers you notice that: $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=7=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$
You can see that for any vectors **a**, **b** and **c**:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c})$$

The products $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ and $\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c})$ are called **scalar triple products**.

This worksheet is one of a series on mathematics produced by the Learning Enhancement Team with funding from the UEA Alumni Fund. Scan the QR-code with a smartphone app for more resources.

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