Learning Enhancement Team

University of East Anglia

STUDENT SUPPORT SERVICE

Steps into Calculus

Basics of Partial Differentiation

This guide introduces the concept of differentiating a function of two variables by using partial differentiation. It will explain what a partial derivative is and how to do partial differentiation.

Introduction

Partial differentiation is used to differentiate functions which have more than one variable in them. It is more general than differentiating functions of one variable, which is known as **ordinary differentiation** and is introduced in the study guide: *What is Differentiation?* Partial differentiation can be used for finding maxima and minima in optimisation and for describing more complicated processes in science in what are known as **partial differential equations** (see the study guide: *Basics of Differential Equations*).

The first functions you encounter are usually functions of one variable $y = f(x)$ as described in the study guide: *Using Functions.* For example the function $y = f(x) = x^2$ takes a value x as its input, squares it and outputs it. The graph of this function is given by the (x, y) coordinates that satisfy the equation $y = x^2$ and the gradient of this function is $dy/dx = 2x$ which comes from differentiating x^2 using the power rule (see the study guide: *Differentiating using the Power Rule*). In other words, the rate of change of y with respect to x will be $2x$. This sort of differentiation is called ordinary differentiation.

Functions can depend on more than one variable. A function with two variables can be written as $z = f(x, y)$ and it has **partial derivatives** with respect to x or y.

> For a function of two variables $z = f(x, y)$:
>
> The partial derivative with respect to x is written as $\dfrac{\partial z}{\partial x}$.
>
> The partial derivative with respect to y is written as $\dfrac{\partial z}{\partial y}$.

Here you can see that partial derivatives are written with a “curly d” which looks like “∂” and not with the Latin letter “d” which is used to write a derivative in functions of one variable. An alternative notation is the subscript notation where:

$$\frac{\partial z}{\partial x} = z_x \qquad \text{and} \qquad \frac{\partial f}{\partial y} = f_y$$

An example of a function with two variables is $z = x^2 + y^2 - 16$ in which *z* depends on both *x* and *y*. You need to supply two inputs in order to get one output. The graph of this function is a three-dimensional surface like a bowl. You could ask, "What is the gradient of this surface?" To find this, do you differentiate with respect to *x* or *y*? Unlike a function of one variable, how the function changes will depend on which variable is altered. Imagine walking down a hill. The gradient is steeper if you walk straight down the hill and less steep if you take the longer route by going down at an angle. In other words, the gradient of the function will depend on which direction you choose. For a function of two variables, you could choose either the *x*-direction or the *y*-direction and the gradient depends on your choice. The gradients are given by the partial derivative with respect to *x* and the partial derivative with respect to *y.*

In this example *z* is a function of two variables *x* and *y* which are independent. Partial differentiation should not be confused with **implicit differentiation** of the implicit function $x^2 + y^2 - 16 = 0$, for example, where *y* is considered to be a function of *x* and therefore not independent of *x*. See study guide: *Implicit Differentiation* for more on this.

Partial differentiation can be applied to functions of more than two variables but, for simplicity, the rest of this study guide deals with functions of two variables, *x* and *y*.

How to do partial differentiation

Partial differentiation builds on the concepts of ordinary differentiation and so you should be familiar with the methods introduced in the *Steps into Calculus* series before you proceed. In fact, for a function of one variable, the partial derivative is the same as the ordinary derivative.

Example: Find the partial derivatives of $z = 3x^2$.

The function is really only a function of *x* and so:

$$\frac{\partial z}{\partial x} = 6x$$

which is the same as the ordinary derivative since $dz/dx = 6x$ too. The function does not depend on *y* and so it does not change as *y* changes and so:

$$\frac{\partial z}{\partial y} = 0$$

In general, however, for a function $z = f(x, y)$:

> $\frac{\partial z}{\partial x}$ is found by keeping *y* constant and differentiating as usual with respect to *x.*
>
> $\frac{\partial z}{\partial y}$ is found by keeping *x* constant and differentiating as usual with respect to *y.*

To help with partial differentiation it is very useful to remember a rule from ordinary differentiation which is discussed in the study guide: *Differentiating using the Power Rule*

> If $y = af(x) + b$ where *a* and *b* are constants, then $\frac{dy}{dx} = af'(x)$

So that constants, such as *b,* when added to functions differentiate to 0 but constants multiplying functions, such as *a,* are retained and so still multiply the derivative.

If you are partially differentiating with respect to *x*, for example, then the variable *y* is kept constant and so must be treated as *a* or *b* in the rules above. Also, since it is being treated as a constant, functions of *y* such as y^2, $8y$ or $\sin(y)$ are also treated as constants and so act as if they were the *a* or *b* in the rule above.

Example: Find the first partial derivatives of $z = x^2 - 6x + y^2 - 4y + 14$.

To find the partial derivative with respect to *x* you can split this into three parts:

$x^2 - 6x$	is a function of *x* only and so partially differentiates to $2x - 6$.
$y^2 - 4y$	is a function of *y* only and so is treated as a constant and so differentiates to 0.
14	is a constant and so differentiates to 0.

And so $\frac{\partial z}{\partial x} = 2x - 6$

To find the partial derivative with respect to *y* you can split this into three parts:

$x^2 - 6x$	is a function of *x* only and so is treated as a constant and so differentiates to 0.
$y^2 - 4y$	is a function of *y* only and so partially differentiates to $2y - 4$.
14	is a constant and so differentiates to 0.

And so $\frac{\partial z}{\partial y} = 2y - 4$

Example: Find the first partial derivatives of $z = x^2 y^3$

To find $\partial z / \partial x$ you must treat *y* as a constant. Then it is the same as ordinary differentiation of the function ax^2 except that the constant *a* is actually y^3 and so:

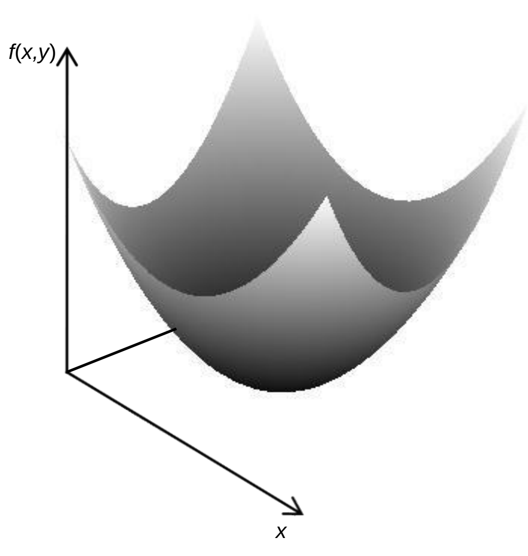
$$\frac{\partial z}{\partial x} = 2xy^3.$$

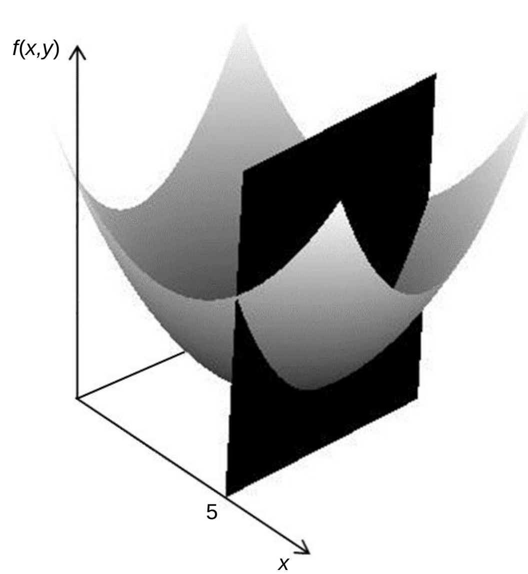
To find $\partial z / \partial y$ you must treat *x* as a constant. Then it is the same as ordinary differentiation of a function such as ay^3 except that the constant *a* is actually x^2 and so:

$$\frac{\partial z}{\partial y} = 3x^2 y^2.$$

Gradients

The graph of the function from the last-but-one example, $z = x^2 - 6x + y^2 - 4y + 14$, is a parabolic surface (a bowl) as shown below.





As you have seen in the previous section a partial derivative is obtained by holding one of the variables constant. If *x* is held constant, say $x = 5$ for example (as shown in the graph on the right), then the function is a quadratic function in *y* only:

$$z = 5^2 - 6 \times 5 + y^2 - 4y + 14 = y^2 - 4y + 9$$

So its graph is a 2D parabola. The graph on the right shows the cross-section of the parabolic surface taken at $x = 5$. Where it intersects the surface is the 2D parabola. The gradient of this parabola is given by $\partial z / \partial y = 2y - 4$ which was calculated in the example from before. If *y* is held constant then the function would become a quadratic in *x* with gradient $\partial z / \partial x = 2x - 6$ as calculated in the second example in this guide.

Higher derivatives

Just as with functions of one variable, functions of many variables can be differentiated more than once to obtain second, third or higher partial derivatives such as:

$$\frac{\partial^2 z}{\partial x^2} \quad \text{or} \quad \frac{\partial^3 z}{\partial y^3}$$

For the function $z = f(x, y)$, partially differentiating the first derivative again using the **partial differential operator** gives second derivatives with respect to *x* and *y:*

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \qquad \text{and} \qquad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$$

These are very similar to the ordinary differentiation case (see study guide: *The Differential Operator* for an explanation of this.)

However, for multivariable functions, these are not the only kinds of second derivative. For multivariable functions it is also possible to partially differentiate with respect to different variables and these are called **mixed derivatives**. So the function $z = f(x, y)$ may be partially differentiated with respect *x* and then *y* or with respect to *y* and then *x:*

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} \qquad \text{and} \qquad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$$

Using the subscript notation, the four second order partial derivatives of *z* can be written as: z_{xx}, z_{yy}, z_{xy} and z_{yx}.

Example: Find all the second order partial derivatives of the function $z = 5x^3y^2$.

First find the first two partial derivatives, $\partial z / \partial x$ and $\partial z / \partial y$ and then partially differentiate these with respect to *x* and *y* to find the second partial derivatives $\partial^2 z / \partial x^2$, $\partial^2 z / \partial y \partial x$, $\partial^2 z / \partial x \partial y$ and $\partial^2 z / \partial y^2$. The diagram below shows how this is done.

Arrows going right are partial differentiation with respect to *y* and arrows going left are partial differentiation with respect to *x*.

$$z = 5x^3y^2$$

$$\frac{\partial z}{\partial x} = 15x^2y^2 \qquad \frac{\partial z}{\partial y} = 10x^3y$$

$$\frac{\partial^2 z}{\partial x^2} = 30xy^2 \qquad \frac{\partial^2 z}{\partial y \partial x} = 30x^2y \quad = \quad \frac{\partial^2 z}{\partial x \partial y} = 30x^2y \qquad \frac{\partial^2 z}{\partial y^2} = 10x^3$$

You can see that the two mixed derivatives are equal. In fact, under certain conditions, this will always be true. This is known as Schwarz' or Young's theorem which states that it does not matter what order the partial differentiation is done.

Want to know more?

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