Learning Enhancement Team

University of East Anglia

STUDENT SUPPORT SERVICE

Model Answers: Basics of Partial Differentiation

These are the model answers for the worksheet that has questions on basics of partial differentiation.

Basics of Partial Differentiation Study Guide

a. The partial derivatives of $z = -2x^{-2}$ are $\frac{\partial z}{\partial x} = 4x^{-3}$ and $\frac{\partial z}{\partial y} = 0$.

Looking at the function $z = -2x^{-2}$ you can tell that this is a function of *x* and so:

$$\frac{\partial z}{\partial x} = -2 \cdot (-2)x^{-2-1} = 4x^{-3}$$

which is the same as the ordinary derivative since $dz/dx = 4x^{-3}$ too.

The function does not depend on *y* and so it does not change as *y* changes and so:

$$\frac{\partial z}{\partial y} = 0$$

b. The partial derivatives of $z = 4e^{y}$ are $\frac{\partial z}{\partial y} = 4e^{y}$ and $\frac{\partial z}{\partial x} = 0$.

Looking at the function $z = 4e^{y}$ you can tell that this is a function of *y* and so:

$$\frac{\partial z}{\partial y} = 4e^{y}$$

which is the same as the ordinary derivative since $dz/dy = 4e^{y}$ too.

The function does not depend on *x* and so it does not change as *x* changes and so:

$$\frac{\partial z}{\partial x} = 0$$

For both of the above questions what you notice is that when the function is just a function of x the partial derivative with respect to x is like the ordinary derivative and the partial derivative with respect to y is zero, since the function doesn't depend on y and so it doesn't change as y changes. Similarly for a function of just y the partial derivative with respect to y is like the ordinary derivative and the partial derivative with respect to x is zero, since the function doesn't depend on x and so it doesn't change as x changes.

2.

a. The first partial derivatives of $z = x^2 + 2x + y^3 - 2y + 6$ are:

$$\frac{\partial z}{\partial x} = 2x + 2 \text{ and } \frac{\partial z}{\partial y} = 3y^2 - 2.$$

To find the partial derivative with respect to x you can split this into three parts:

$x^2 + 2x$	is a function of x only and so partially differentiates to $2x + 2$.
$y^3 - 2y$	is a function of y only and so is treated as a constant and so differentiates to 0.
6	is a constant and so differentiates to 0.

And so $\frac{\partial z}{\partial x} = 2x + 2$

To find the partial derivative with respect to y you can split this into three parts:

$x^2 + 2x$	is a function of x only and so is treated as a constant and so differentiates to 0.
$y^3 - 2y$	is a function of y only and so partially differentiates to $3y^2 - 2$.
6	is a constant and so differentiates to 0.

And so $\frac{\partial z}{\partial y} = 3y^2 - 2$

b. The first partial derivatives of $z = 3x^3 - 2y^3 + x^{-2} - \frac{y^{-2}}{4} + 2016$ are:

$$\frac{\partial z}{\partial x} = 9x^2 - 2x^{-3} \text{ and } \frac{\partial z}{\partial y} = -6y^2 + \frac{y^{-3}}{2}.$$

To find the partial derivative with respect to x you can split this into three parts:

$3x^3 + x^{-2}$	is a function of x only and so partially differentiates to $9x^2 - 2x^{-3}$.
$-2y^3 - \frac{y^{-2}}{4}$	is a function of y only and so is treated as a constant and so differentiates to 0.
2016	is a constant and so differentiates to 0.

And so $\frac{\partial z}{\partial x} = 9x^2 - 2x^{-3}$.

To find the partial derivative with respect to y you can split this into three parts:

$3x^3 + x^{-2}$	is a function of x only and so is treated as a constant and so differentiates to 0.
$-2y^3 - \frac{y^{-2}}{4}$	is a function of y only and so differentiates to $-6y^2 + \frac{y^{-3}}{2}$.
2016	is a constant and so differentiates to 0.

And so $\frac{\partial z}{\partial y} = -6y^2 + \frac{y^{-3}}{2}$.

c. The first partial derivatives of $z = xy^2$ are $\frac{\partial z}{\partial x} = y^2$ and $\frac{\partial z}{\partial y} = 2xy$.

To find $\partial z / \partial x$ you must treat y as a constant. Then it is the same as ordinary differentiation of the function ax except that the constant a is actually y^2 and so:

$$\frac{\partial z}{\partial x} = y^2 .$$

To find $\partial z / \partial y$ you must treat x as a constant. Then it is the same as ordinary differentiation of the function ay^2 except that the constant a is actually x and so:

$$\frac{\partial z}{\partial y} = 2xy .$$

d. The first partial derivatives of $z = \frac{x + y}{x^2 + 2y}$ are:

$$\frac{\partial z}{\partial x} = \frac{-x^2 + 2y - 2xy}{\left(x^2 + 2y\right)^2} \text{ and } \frac{\partial z}{\partial y} = \frac{x^2 - 2x}{\left(x^2 + 2y\right)^2} .$$

To find $\partial z / \partial x$ you must treat *y* as a constant. Then it is the same as ordinary differentiation of the function $\frac{x+a}{x^2+2a}$ except that the constant *a* is actually *y* and so you use the quotient rule (To remind yourself of the rule you can see the study guide: *The Quotient Rule*):

$$\frac{\partial z}{\partial x} = \frac{1 \cdot (x^2+2y) - (x+y) \cdot 2x}{(x^2+2y)^2} = \frac{-x^2+2y-2xy}{(x^2+2y)^2}$$

To find $\partial z / \partial y$ you must treat *x* as a constant. Then it is the same as ordinary differentiation of the function $\frac{a+y}{a^2+2y}$ except that the constant *a* is actually *x* and so you use the quotient rule (To remind yourself of the rule you can see the study guide: *The Quotient Rule*):

$$\frac{\partial z}{\partial y} = \frac{1 \cdot (x^2+2y) - (x+y) \cdot 2}{(x^2+2y)^2} = \frac{x^2-2x}{(x^2+2y)^2}.$$

e. The first partial derivatives of $z = xe^y + y^2\cos(x) + y\ln(x)$ are:

$$\frac{\partial z}{\partial x} = e^y - y^2\sin(x) + \frac{y}{x} \text{ and } \frac{\partial z}{\partial y} = xe^y + 2y\cos(x) + \ln(x).$$

To find $\partial z / \partial x$ you must treat *y* as a constant. Then it is the same as ordinary differentiation of the function $z = xe^a + a^2\cos(x) + a\ln(x)$ except that the constant *a* is actually *y* and so:

$$\frac{\partial z}{\partial x} = e^y - y^2\sin(x) + \frac{y}{x}$$

To find $\partial z / \partial y$ you must treat *x* as a constant. Then it is the same as ordinary differentiation of a function such as $z = ae^y + y^2\cos(a) + y\ln(a)$ except that the constant *a* is actually *x* and so:

$$\frac{\partial z}{\partial y} = xe^y + 2y\cos(x) + \ln(x)$$

f. The first partial derivatives of $z = xe^{x+y} - ye^{x-y}$ are:

$$\frac{\partial z}{\partial x} = e^{x+y}(1+x) - ye^{x-y} \text{ and } \frac{\partial z}{\partial y} = xe^{x+y} + e^{x-y}(y-1).$$

To find $\partial z/\partial x$ you must treat *y* as a constant. Then it is the same as ordinary differentiation of the function $z = xe^{x+a} - ae^{x-a}$ except that the constant *a* is actually *y* and so you use the product rule (To remind yourself of the rule you can see the study guide: *The Product Rule*):

$$\frac{\partial z}{\partial x} = 1 \cdot e^{x+y} + xe^{x+y} - ye^{x-y} = e^{x+y}(1+x) - ye^{x-y}$$

To find $\partial z/\partial y$ you must treat *x* as a constant. Then it is the same as ordinary differentiation of a function such as $z = ae^{a+y} - ye^{a-y}$ except that the constant *a* is actually *x* and so you use the product rule (To remind yourself of the rule you can see the study guide: *The Product Rule*):

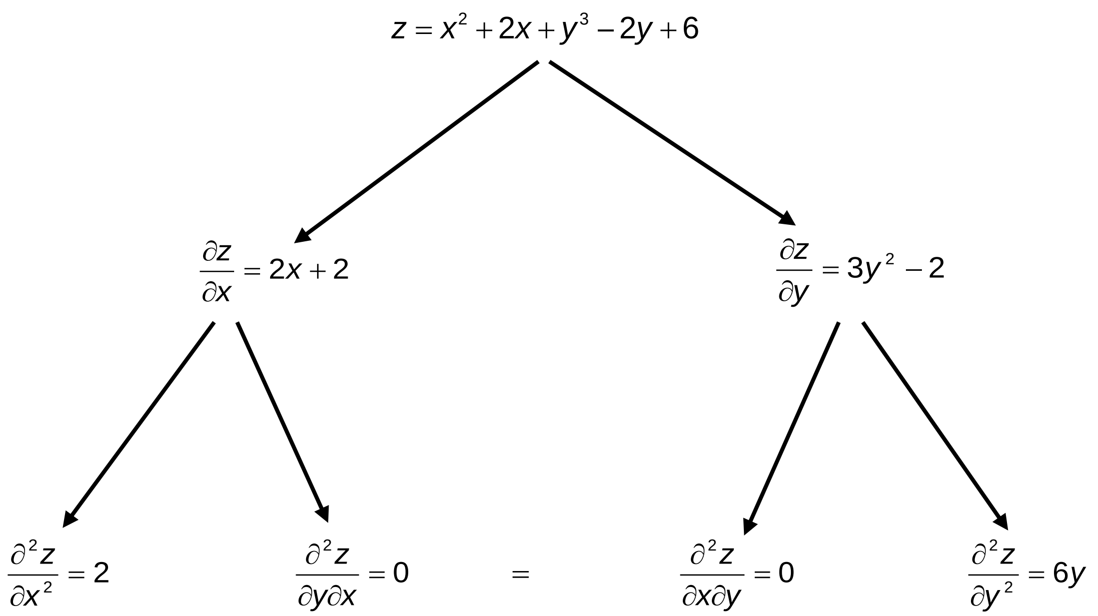
$$\frac{\partial z}{\partial y} = xe^{x+y} - \left(e^{x-y} - ye^{x-y}\right) = xe^{x+y} - e^{x-y} + ye^{x-y} = xe^{x+y} + e^{x-y}(y-1)$$

3.

a. The second order partial derivatives of $z = x^2 + 2x + y^3 - 2y + 6$ are:

$$\frac{\partial^2 z}{\partial x^2} = 2, \quad \frac{\partial^2 z}{\partial y^2} = 6y \quad \text{and} \quad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} = 0$$

First find the first two partial derivatives, $\partial z/\partial x$ and $\partial z/\partial y$ and then partially differentiate these with respect to *x* and *y* to find the second partial derivatives $\partial^2 z/\partial x^2$, $\partial^2 z/\partial y\partial x$, $\partial^2 z/\partial x\partial y$ and $\partial^2 z/\partial y^2$. The diagram below shows how this is done. Arrows going right are partial differentiation with respect to *y* and arrows going left are partial differentiation with respect to *x*.



You can see that the two mixed derivatives are equal.

b. The second order partial derivatives of $z = 3x^3 - 2y^3 + x^{-2} - \frac{y^{-2}}{4} + 2016$ are:

$$\frac{\partial^2 z}{\partial x^2} = 18x + 6x^{-4}, \quad \frac{\partial^2 z}{\partial y^2} = -12y - \frac{3}{2}y^{-4} \text{ and } \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} = 0.$$

First find the first two partial derivatives, $\partial z / \partial x$ and $\partial z / \partial y$ and then partially differentiate these with respect to *x* and *y* to find the second partial derivatives $\partial^2 z / \partial x^2$, $\partial^2 z / \partial y \partial x$, $\partial^2 z / \partial x \partial y$ and $\partial^2 z / \partial y^2$. The diagram below shows how this is done. Arrows going right are partial differentiation with respect to *y* and arrows going left are partial differentiation with respect to *x*.

$$z = 3x^3 - 2y^3 + x^{-2} - \frac{y^{-2}}{4} + 2016$$

$$\frac{\partial z}{\partial x} = 9x^2 - 2x^{-3} \qquad \frac{\partial z}{\partial y} = -6y^2 + \frac{y^{-3}}{2}$$

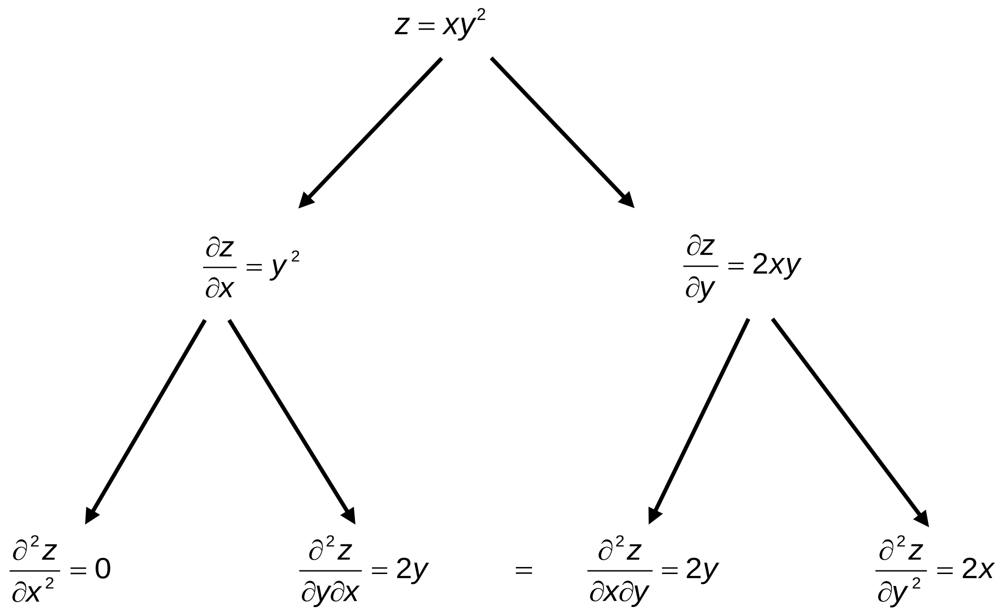
$$\frac{\partial^2 z}{\partial x^2} = 18x + 6x^{-4} \qquad \frac{\partial^2 z}{\partial y \partial x} = 0 \quad = \quad \frac{\partial^2 z}{\partial x \partial y} = 0 \qquad \frac{\partial^2 z}{\partial y^2} = -12y - \frac{3}{2}y^{-4}$$

You can see that the two mixed derivatives are equal.

c. The second order partial derivatives of $z = xy^2$ are:

$$\frac{\partial^2 z}{\partial x^2} = 0, \ \frac{\partial^2 z}{\partial y^2} = 2x \ \text{and} \ \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} = 2y.$$

First find the first two partial derivatives, $\partial z / \partial x$ and $\partial z / \partial y$ and then partially differentiate these with respect to x and y to find the second partial derivatives $\partial^2 z / \partial x^2$, $\partial^2 z / \partial y \partial x$, $\partial^2 z / \partial x \partial y$ and $\partial^2 z / \partial y^2$. The diagram below shows how this is done. Arrows going right are partial differentiation with respect to y and arrows going left are partial differentiation with respect to x.



You can see that the two mixed derivatives are equal.

d. The second order partial derivatives of $z = \dfrac{x+y}{x^2+2y}$ are:

$$\frac{\partial^2 z}{\partial x^2} = \frac{2\left(x^3 + 3x^2 y - 6xy - 2y^2\right)}{\left(x^2+2y\right)^3},$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{-4\left(x^2 - 2x\right)}{\left(x^2+2y\right)^3} \text{ and}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} = \frac{-2\left(x^3 - 3x^2 - 2xy + 2y\right)}{\left(x^2+2y\right)^3}$$

First find the first two partial derivatives, $\partial z / \partial x$ and $\partial z / \partial y$ and then partially differentiate these with respect to *x* and *y* to find the second partial derivatives $\partial^2 z / \partial x^2$, $\partial^2 z / \partial y \partial x$, $\partial^2 z / \partial x \partial y$ and $\partial^2 z / \partial y^2$. The diagrams below shows how this is done. You can turn your paper in the horizontal direction to have all the partial derivatives in one diagram. Arrows going right are partial differentiation with respect to *y* and arrows going left are partial differentiation with respect to *x*.

$$z = \frac{x+y}{x^2+2y} \longrightarrow \frac{\partial z}{\partial x} = \frac{-x^2 + 2y - 2xy}{\left(x^2+2y\right)^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{2\left(x^3 + 3x^2 y - 6xy - 2y^2\right)}{\left(x^2+2y\right)^3} \qquad \frac{\partial^2 z}{\partial y \partial x} = \frac{-2\left(x^3 - 3x^2 - 2xy + 2y\right)}{\left(x^2+2y\right)^3}$$

$$z = \frac{x+y}{x^2+2y} \longrightarrow \frac{\partial z}{\partial y} = \frac{x^2 - 2x}{\left(x^2+2y\right)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-2\left(x^3 - 3x^2 - 2xy + 2y\right)}{\left(x^2+2y\right)^3} \qquad \frac{\partial^2 z}{\partial y^2} = \frac{-4\left(x^2 - 2x\right)}{\left(x^2+2y\right)^3}$$

You can see that the two mixed derivatives are equal.

e. The second order partial derivatives of $z = xe^{y} + y^{2}\cos(x) + y\ln(x)$ are:

$$\frac{\partial^2 z}{\partial x^2} = -y^2\cos(x) - \frac{y}{x^2},$$

$$\frac{\partial^2 z}{\partial y^2} = xe^{y} + 2\cos(x) \text{ and}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} = e^{y} - 2y\sin(x) + \frac{1}{x}$$

First find the first two partial derivatives, $\partial z/\partial x$ and $\partial z/\partial y$ and then partially differentiate these with respect to *x* and *y* to find the second partial derivatives $\partial^2 z/\partial x^2$, $\partial^2 z/\partial y\partial x$, $\partial^2 z/\partial x\partial y$ and $\partial^2 z/\partial y^2$. The diagrams below shows how this is done. You can turn your paper in the horizontal direction to have all the partial derivatives in one diagram. Arrows going right are partial differentiation with respect to *y* and arrows going left are partial differentiation with respect to *x*.

$$z = xe^{y} + y^2\cos(x) + y\ln(x) \longrightarrow \frac{\partial z}{\partial x} = e^{y} - y^2\sin(x) + \frac{y}{x}$$

$$\swarrow \qquad \searrow$$

$$\frac{\partial^2 z}{\partial x^2} = -y^2\cos(x) - \frac{y}{x^2} \qquad \frac{\partial^2 z}{\partial y \partial x} = e^{y} - 2y\sin(x) + \frac{1}{x}$$

$$z = xe^{y} + y^2\cos(x) + y\ln(x) \longrightarrow \frac{\partial z}{\partial y} = xe^{y} + 2y\cos(x) + \ln(x)$$

$$\swarrow \qquad \searrow$$

$$\frac{\partial^2 z}{\partial x \partial y} = e^{y} - 2y\sin(x) + \frac{1}{x} \qquad \frac{\partial^2 z}{\partial y^2} = xe^{y} + 2\cos(x)$$

You can see that the two mixed derivatives are equal.

f. The second order partial derivatives of $z = xe^{x+y} - ye^{x-y}$ are:

$$\frac{\partial^2 z}{\partial x^2} = e^{x+y}(x+2) - ye^{x-y},$$

$$\frac{\partial^2 z}{\partial y^2} = xe^{x+y} + e^{x-y}(2-y) \text{ and}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} = e^{x+y}(x+1) + e^{x-y}(y-1)$$

First find the first two partial derivatives, $\partial z / \partial x$ and $\partial z / \partial y$ and then partially differentiate these with respect to *x* and *y* to find the second partial derivatives $\partial^2 z / \partial x^2$, $\partial^2 z / \partial y \partial x$, $\partial^2 z / \partial x \partial y$ and $\partial^2 z / \partial y^2$. The diagrams below shows how this is done. You can turn your paper in the horizontal direction to have all the partial derivatives in one diagram. Arrows going right are partial differentiation with respect to *y* and arrows going left are partial differentiation with respect to *x*.

$$z = xe^{x+y} - ye^{x-y} \longrightarrow \frac{\partial z}{\partial x} = e^{x+y}(1+x) - ye^{x-y}$$

$$\swarrow \qquad \searrow$$

$$\frac{\partial^2 z}{\partial x^2} = e^{x+y}(x+2) - ye^{x-y} \qquad \frac{\partial^2 z}{\partial y \partial x} = e^{x+y}(x+1) + e^{x-y}(y-1)$$

$$z = xe^{x+y} - ye^{x-y} \longrightarrow \frac{\partial z}{\partial y} = xe^{x+y} + e^{x-y}(y-1)$$

$$\swarrow \qquad \searrow$$

$$\frac{\partial^2 z}{\partial x \partial y} = e^{x+y}(x+1) + e^{x-y}(y-1) \qquad \frac{\partial^2 z}{\partial y^2} = xe^{x+y} + e^{x-y}(2-y)$$

You can see that the two mixed derivatives are equal.