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STUDENT SUPPORT SERVICE

Implicit Differentiation

This guide introduces differentiation of implicit functions by application of the differential operator.

Introduction: The differential operator

The study guide: *The Differential Operator* describes how you can perform differentiation by the application of an operator called **the differential operator**. It is strongly recommended that you read this guide before proceeding.

The differential operator with respect to x is written as either $\frac{d}{dx}$ or D_x.

You can use the differential operator to find the derivative of y if it is a function of x. You can also use the operator on functions of y too, as they are composite functions of x. The following discussion assumes that y is a function of x. (If y is not a function of x you need **partial differentiation** to perform differentiation, see study guide: *Basics of Partial Differentiation*.) You can use the differential operator to differentiate the five basic functions of y, as shown in the following table.

Function of y	Derivative
ay^n	$any^{n-1}\frac{dy}{dx}$
$a\sin ky$	$ak\cos ky\frac{dy}{dx}$
$a\cos ky$	$-ak\sin ky\frac{dy}{dx}$
ae^{ky}	$ake^{ky}\frac{dy}{dx}$
$a\ln(ky)$	$\frac{a}{y}\frac{dy}{dx}$

You can see that the derivatives follow a similar pattern to those of the basic functions of x (see study guide: *Differentiating Basic Functions*) but the application of the **chain rule form of the differential operator**, results in multiplication by dy/dx too.

Differentiating implicit functions

All functions which only contain two variables, such as *x* and *y*, can be written as $g(x,y)=0$.

An **implicit function** of *x* and *y* is written as $g(x,y)=0$.

Some of these functions can be rearranged so that *y* can be expressed solely as a function of *x*:

It is said that *y* is an **explicit** function of *x* if you can write $y=f(x)$.

Most of the functions discussed in the *Steps into Calculus* study guides are of this form. However you cannot use basic methods to write all functions $g(x,y)=0$ as $y=f(x)$.

Finding the derivative of the function $g(x,y)=0$ can be achieved one of two ways:

(i) If you can rearrange $g(x,y)=0$ for *y*, you get $y=f(x)$ and you can use the basic rules of differentiation to find the derivative.

(ii) If you cannot rearrange $g(x,y)=0$ for *y* using basic methods you need to apply the differential operator to find the derivative, this is called **implicit differentiation**.

Example: Differentiate $y^3+\sin x+y=0$ with respect to *x*.

As you cannot use basic methods to rearrange this function to get $y=f(x)$, you must differentiate implicitly. Firstly apply the differential operator to each term:

$$\frac{d}{dx}\left(y^3\right)+\frac{d}{dx}(\sin x)+\frac{d}{dx}(y)=\frac{d}{dx}(0)$$

The first term on the left requires the chain rule form of the differential operator as y^3 is a composite function of *x*. The rules on the first page of this guide tell you that:

$$\frac{d}{dx}\left(y^3\right)=3y^2\frac{dy}{dx}$$

The second term on the left is a basic function of *x* and so:

$$\frac{d}{dx}(\sin x) = \cos x$$

The third term is the derivative of *y* with respect to *x*:

$$\frac{d}{dx}(y) = \frac{dy}{dx}$$

The right hand side is the derivative of 0 with respect to *x* (which is zero). Combining these results together gives the derivative of $y^3 + \sin x + y = 0$ as:

$$3y^2\frac{dy}{dx} + \cos x + \frac{dy}{dx} = 0$$

This result is fine as it is written but you can rearrange it for dy/dx. Firstly subtract $\cos x$ from each side to give:

$$3y^2\frac{dy}{dx} + \frac{dy}{dx} = -\cos x$$

Now each term on the left hand side is multiplied by dy/dx and so you can factorise:

$$(3y^2 + 1)\frac{dy}{dx} = -\cos x$$

Finally dividing by $3y^2 + 1$ gives:

$$\frac{dy}{dx} = \frac{-\cos x}{3y^2 + 1}$$

Note that the derivative is a function of both *y* and *x*, this is very common when you use implicit differentiation.

Example: Differentiate $x^3y^2 + e^y - 3x = 0$ with respect to *x*.

As you cannot rearrange this function into the form $y = f(x)$ using basic methods, you must use implicit differentiation. Application of the differential operator gives:

$$\frac{d}{dx}(x^3y^2) + \frac{d}{dx}(e^y) - \frac{d}{dx}(3x) = \frac{d}{dx}(0)$$

The first term is a product of two functions and so you use the product rule:

$$\frac{d}{dx}(x^3y^2) = x^3\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x^3)$$

As y^2 is a composite function of *x*, the rules on the first page of this guide tell you that:

$$\frac{d}{dx}\left(y^2x^3\right) = x^3\left(2y\frac{dy}{dx}\right) + y^2\left(3x^2\right) = 2x^3y\frac{dy}{dx} + 3x^2y^2$$

The second term involves differentiating e^y which is a composite function of x. The rules on the first page of this guide tell you that:

$$\frac{d}{dx}\left(e^y\right) = e^y\frac{dy}{dx}$$

The derivative of the third term is -3 and so, combining these results gives:

$$2x^3y\frac{dy}{dx} + 3x^2y^2 + e^y\frac{dy}{dx} - 3 = 0$$

You can rearrange for dy/dx by subtracting $3x^2y^2$ from and adding 3 to each side, then factorising the left hand side and finally dividing by $2x^3y + e^y$ to find that:

$$\frac{dy}{dx} = \frac{3 - 3x^2y^2}{2x^3y + e^y}$$

This is only defined when $2x^3y + e^y \neq 0$ as dividing by zero is not allowed. The deeper implications of this are not obvious, ask a **Learning Enhancement Tutor** if you want to know more.

Want to know more?

If you have any further questions about this topic you can make an appointment to see a **Learning Enhancement Tutor** in the **Student Support Service**, as well as speaking to your lecturer or adviser.

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