

Argand Diagrams and Polar Form

This guide introduces Argand diagrams which are used to visualise complex numbers. It also shows how to calculate the modulus and argument of a complex number, their role in the polar form of a complex number and how to convert between Cartesian and polar forms.

Introduction

One of the problems that mathematicians came across when they were exploring the idea of complex numbers was their visualisation. A complex number does not appear anywhere on a real number line and so would not appear in coordinates in the xy -plane. This problem was solved in an ingenious way in the 19th century by Jean-Robert Argand (building on the work of Caspar Wessel). An **Argand diagram** uses the **real** and **imaginary parts** of a complex number as analogues of x and y in the Cartesian plane. The area of an Argand diagram is called the **complex plane** by mathematicians.

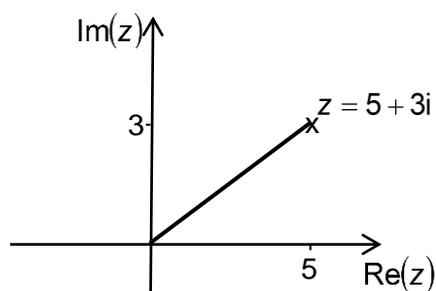
The study guide: [Basics of Complex Numbers](#) describes the **Cartesian form** of a complex number z as:

$$z = a + bi$$

Cartesian form of a complex number

Where a is the **real part** of a complex number, written $\text{Re}(z)$, and b is the **imaginary part** of a complex number, written $\text{Im}(z)$. In an Argand diagram the horizontal axis defines the real part of the complex number and the vertical axis defines the imaginary part. A complex number in Cartesian form has the coordinate (a, b) in an Argand diagram. A purely real number is positioned on the horizontal axis of an Argand diagram and a purely imaginary number is positioned on the vertical axis of an Argand diagram.

Example: Represent the complex number $z = 5 + 3i$ on an Argand diagram.



The position of the number $z = 5 + 3i$ is shown as an "x" but the number itself is usually represented as a line from the origin to the point.

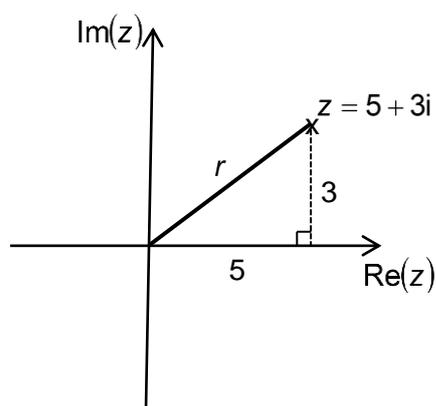
You should label the horizontal axis as $\text{Re}(z)$ and the vertical axis as $\text{Im}(z)$ in an Argand diagram.

The modulus and argument of a complex number

As soon as you represent a complex number visually you can begin to explore other ways of describing it. For example, an Argand diagram shows the similarity of a complex number with a vector, you can think of $\text{Re}(z)$ and $\text{Im}(z)$ being the number of steps along the real and imaginary axes respectively required to 'reach' z .

You can also get to z in a straight line directly from the origin. In order to do this all you need to know is what direction to face and how far to go. How far you need to go is the length of the line representing the complex number in an Argand diagram and is called the **modulus of the complex number**. It is very common to represent the modulus of a complex number z by the symbol $|z|$ or simply by the letter r . The value of the modulus is always positive.

You can use Pythagoras' theorem to find the modulus of a complex number (see study guide: [Pythagoras' Theorem](#)). Below, the Argand given for $z = 5 + 3i$ from page 1 of the guide has been annotated to reveal a right-angled triangle:



As you can see, the modulus of the complex number is the **hypotenuse** of the right-angled triangle. Using Pythagoras' theorem:

$$|z|^2 = r^2 = 5^2 + 3^2 = 34$$

so $|z| = r = \sqrt{34}$ (taking the positive root)

The modulus of a general complex number written in Cartesian form as $z = a + bi$ is:

$$|z| = r = \sqrt{a^2 + b^2}$$

Example: Find the modulus of the following complex numbers:

(a) $z = -2 - i$ (b) $z = -6i$ (c) $z = 3.4$

(a) Here $a = -2$ and $b = -1$ and so $|z| = \sqrt{(-2)^2 + (-1)^2} = \sqrt{5}$

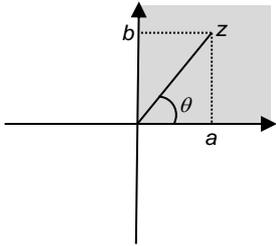
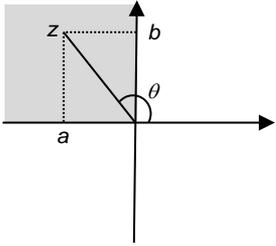
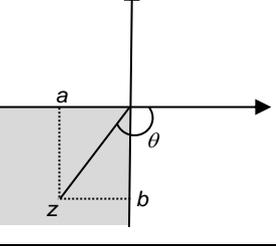
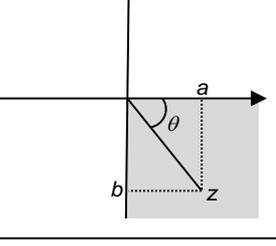
(b) Here $a = 0$ and $b = -6$ and so $|z| = \sqrt{0^2 + (-6)^2} = 6$

(c) Here $a = 3.4$ and $b = 0$ and so $|z| = \sqrt{3.4^2 + 0^2} = 3.4$

The modulus tells you how far to go from the origin to get to a complex number in an Argand diagram. You also need to know what direction the complex number is. This is known as the **argument** of a complex number which is written either as $\text{Arg}(z)$ or the Greek letter θ .

Arguments are **always measured in radians**
and take values in the range $-\pi < \theta \leq \pi$

Complex numbers either **on the real axis or in the upper half of the complex plane** have a **positive argument** measured anti-clockwise from the real axis. Complex numbers in the **lower half of the complex plane** have a **negative argument** measured clockwise from the real axis. You can use the inverse tangent (or arctan) function to find arguments. This can be confusing and the table below is designed to help you. It is a very good idea to sketch your complex number before trying to calculate its argument.

Quadrant (shaded)	Argand diagram sketch	Argument	
		Information	Formula
1		a is positive b is positive argument is positive	$\theta = \tan^{-1}\left(\frac{b}{a}\right)$
2		a is negative b is positive argument is positive	$\theta = \pi + \tan^{-1}\left(\frac{b}{a}\right)$
3		a is negative b is negative argument is negative	$\theta = -\pi + \tan^{-1}\left(\frac{b}{a}\right)$
4		a is positive b is negative argument is negative	$\theta = \tan^{-1}\left(\frac{b}{a}\right)$

The real axis is part of quadrants 1 and 2 (not quadrants 3 and 4). Given this:

Positive real numbers are in quadrant 1 and have an argument of 0.

Negative real numbers are in quadrant 2 and have an argument of π .

Also:

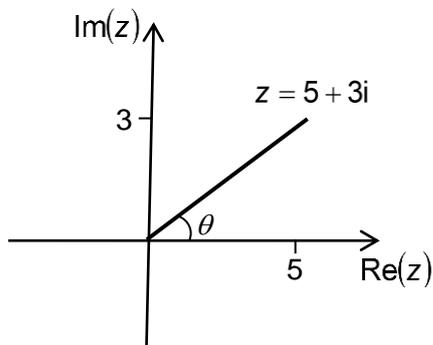
The argument of a purely imaginary number above the real axis (positive b) is $\pi/2$.

The argument of a purely imaginary number below the real axis (negative b) is $-\pi/2$.

Example: Find the modulus of the following complex numbers:

- (a) $z = 5 + 3i$ (b) $z = -2 - i$ (c) $z = -6i$ (d) $z = 3.4$

(a)



As you can see, $z = 5 + 3i$ is in quadrant 1.

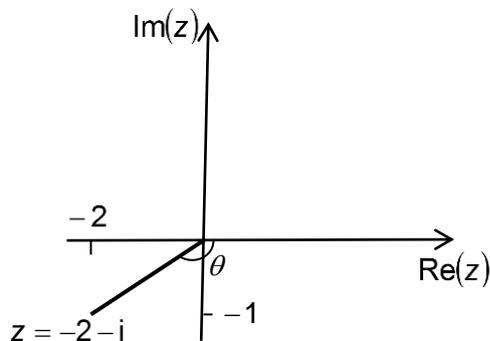
Also $\text{Re}(z) = a = 5$ and $\text{Im}(z) = b = 3$.

Using the appropriate formula from the table:

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{3}{5}\right) = 0.54 \text{ to 2 decimal places}$$

So for $z = 5 + 3i$, $\theta = \text{Arg}(z) = 0.54$ radians.

(b)



As you can see, $z = -2 - i$ is in quadrant 3.

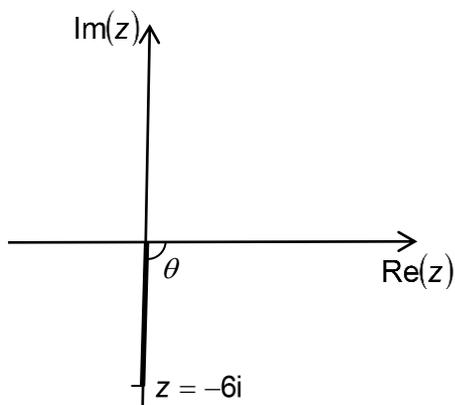
Also $\text{Re}(z) = a = -2$ and $\text{Im}(z) = b = -1$.

Using the appropriate formula from the table:

$$\theta = -\pi + \tan^{-1}\left(\frac{b}{a}\right) = -\pi + \tan^{-1}\left(\frac{-1}{-2}\right) = -2.68 \text{ to 2 d.p.}$$

So for $z = -2 - i$, $\theta = \text{Arg}(z) = -2.68$ radians.

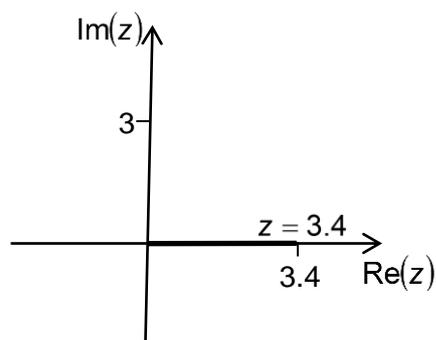
(c)



As you can see, $z = -6i$ is a purely imaginary number below the real axis with $\text{Re}(z) = a = 0$ and $\text{Im}(z) = b = -6$.

This means that $\theta = \text{Arg}(z) = -\pi/2$ as $z = -6i$ is at a right-angle to the real axis.

(d)



As you can see, $z = 3.4$ is a positive, purely real number with $\text{Re}(z) = a = 3.4$ and $\text{Im}(z) = b = 0$.

Positive, real numbers are defined as being in quadrant 1.

This means that $\theta = \text{Arg}(z) = 0$.

Polar form of a complex number

As you have seen, a complex number can be defined not only in terms of its real and imaginary parts but also in terms of its modulus and argument. Just as the Cartesian form of a complex number uses $\text{Re}(z)$ and $\text{Im}(z)$ there is another way of writing a complex number which uses $|z|$ and $\text{Arg}(z)$ called the **polar form**. By definition the polar form of a complex number is:

$$z = re^{i\theta}$$

where r is the modulus and θ is the argument of z

here e is the base of natural logarithm (see study guide: [Exponential Functions](#) for more details). The derivation of this form of a complex number is beyond the scope of this guide but a [Learning Enhancement Tutor](#) will be happy to explain it to you. The polar form is simply another way of writing a complex number, and you need to be comfortable converting from Cartesian form to polar form and vice-versa. The previous sections have shown you how to calculate the modulus and argument of complex number if you know the real and imaginary parts and so you can use those methods to help you convert a complex number from Cartesian form to polar form.

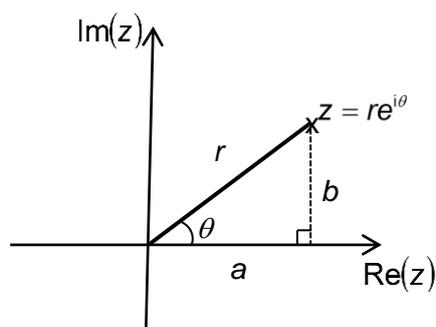
Example: Write the following complex numbers in polar form:

- (a) $z = 5 + 3i$ (b) $z = -2 - i$ (c) $z = -6i$ (d) $z = 3.4$

You can use the results from earlier in this guide to work out the polar form.

- (a) For $z = 5 + 3i$, $r = \sqrt{34}$ and $\theta = 0.54$ and so the polar form is $z = \sqrt{34}e^{0.54i}$.
- (b) For $z = -2 - i$, $r = \sqrt{5}$ and $\theta = -2.68$ and so the polar form is $z = \sqrt{5}e^{-2.68i}$.
- (c) For $z = -6i$, $r = 6$ and $\theta = -\pi/2$ and so the polar form is $z = 6e^{\frac{\pi}{2}i}$.
- (d) For $z = 3.4$, $r = 3.4$ and $\theta = 0$ and so the polar form is $z = 3.4e^0 = 3.4$ as any number to the power 0 is equal to 1.

You now need to know how to convert a complex number in polar form to Cartesian form. To do this you also use trigonometry. Look at the Argand diagram below.



You can use trigonometric ratios to find a and b in terms of r and θ as you have a right-angled triangle. Here $\text{Re}(z) = a = r \cos \theta$ and $\text{Im}(z) = b = r \sin \theta$ and so you can write the Cartesian form of a complex number as:

$$z = r \cos \theta + r \sin \theta i$$

This is an important result that links complex numbers and trigonometry.

You can read the study guide: [Trigonometric Ratios: Sine, Cosine and Tangent](#) for more details about the definition of sine and cosine in a right-angled triangle.

Example: Write the following complex numbers in Cartesian form:

(a) $z = 2e^{1.2i}$ (b) $z = \sqrt{3}e^{-1.4i}$ (c) $z = e^{i\pi}$

(a) For $z = 2e^{1.2i}$, $r = 2$ and $\theta = 1.2$ to convert to Cartesian form use:

$\operatorname{Re}(z) = a = r \cos \theta$: so (using radians) $a = 2 \cos 1.2 = 0.72$ to 2 decimal places.

$\operatorname{Im}(z) = b = r \sin \theta$: so (using radians) $b = 2 \sin 1.2 = 1.86$ to 2 decimal places.

So the Cartesian form of $z = 2e^{1.2i}$ is $z = 0.72 + 1.86i$.

(b) For $z = \sqrt{3}e^{-1.4i}$, $r = \sqrt{3}$ and $\theta = -1.4$ to convert to Cartesian form use:

$\operatorname{Re}(z) = a = r \cos \theta$: so (using radians) $a = \sqrt{3} \cos(-1.4) = 0.29$ to 2 decimal places.

$\operatorname{Im}(z) = b = r \sin \theta$: so (using radians) $b = \sqrt{3} \sin(-1.4) = -1.71$ to 2 decimal places.

So the Cartesian form of $z = \sqrt{3}e^{-1.4i}$ is $z = 0.29 - 1.71i$.

(c) For $z = e^{i\pi}$, $r = 1$ and $\theta = \pi$ to convert to Cartesian form use:

$\operatorname{Re}(z) = a = r \cos \theta$ and so $a = \cos \pi = -1$.

$\operatorname{Im}(z) = b = r \sin \theta$ and so $b = \sin \pi = 0$.

So the Cartesian form of $z = e^{i\pi}$ is $z = -1$. This is an important result and is discussed in the study guide: *Euler's Identity and De Moivre's Theorem*.

Want to know more?

If you have any further questions about this topic you can make an appointment to see a [Learning Enhancement Tutor](#) in the [Student Support Service](#), as well as speaking to your lecturer or adviser.

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