Learning Enhancement Team

University of East Anglia

STUDENT SUPPORT SERVICE

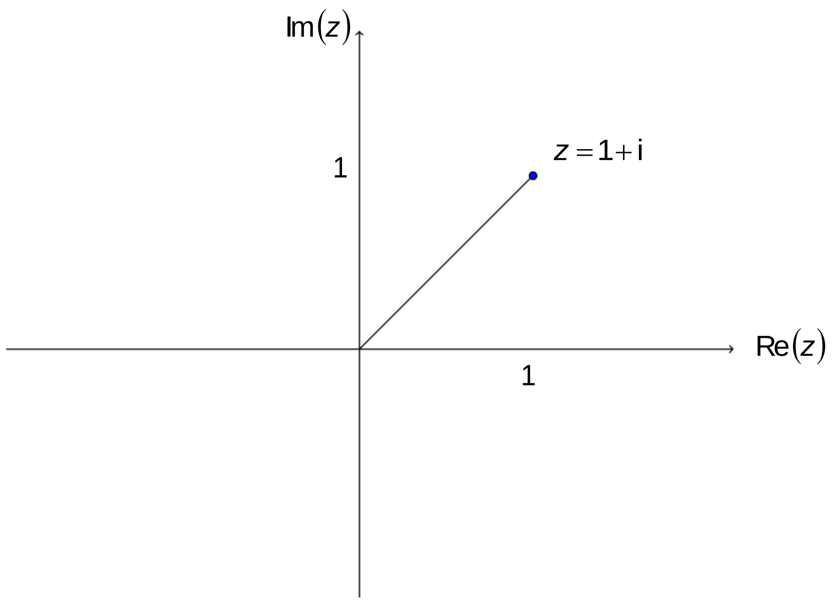
Model answers: Argand Diagrams and Polar Form

Argand Diagrams and Polar Form study guide

Remember that you should write the argument of a complex number using radians.

1. i) For $z = 1 + \mathrm{i}$, $|z| = \sqrt{2}$, $\mathrm{Arg}(z) = \pi/4$ and so $z = \sqrt{2}e^{\frac{\pi}{4}\mathrm{i}}$ in polar form.

The Argand diagram for $z = 1 + \mathrm{i}$ is given below.



You can find the modulus using the formula from the study guide. Here, $a = b = 1$ and so:

$$|z| = \sqrt{(1)^2 + (1)^2}$$
$$|z| = \sqrt{1+1}$$

$|z| = \sqrt{2}$

You can see from your Argand diagram that $z = 1 + \mathrm{i}$ is in quadrant 1. Remember from the study guide that the argument of a complex number *z* in quadrant 1 is given by the formula:

$$\text{Arg}(z) = \tan^{-1}\left(\frac{b}{a}\right).$$

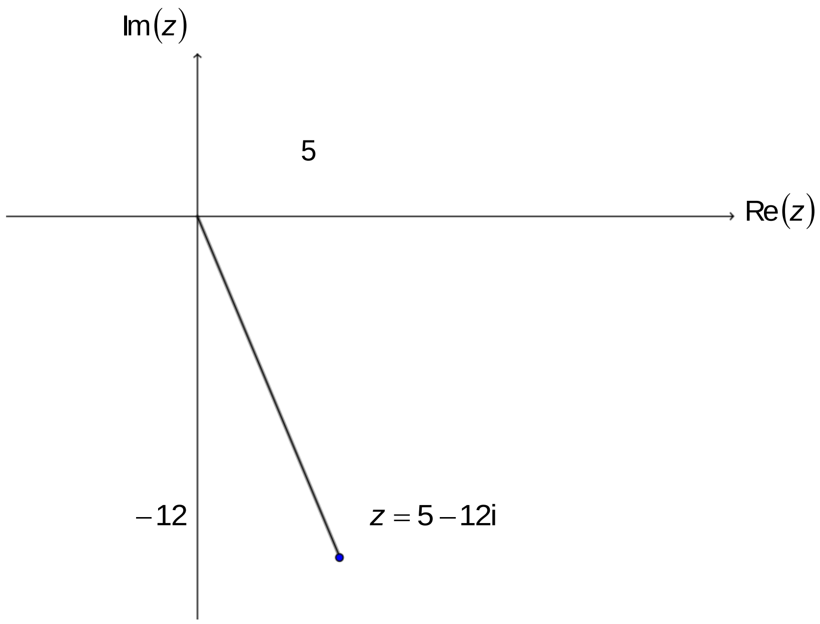
Here, $a = b = 1$ and so you can see that:

$$\text{Arg}(z) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

So you can write $z = 1 + \mathrm{i}$ in polar form as $z = \sqrt{2}e^{\frac{\pi}{4}\mathrm{i}}$.

ii) For $z = 5 - 12\mathrm{i}$, $|z| = 13$, $\text{Arg}(z) = -1.176$ and so $z = 13e^{-1.176\,\mathrm{i}}$ in polar form.

The Argand diagram for $z = 5 - 12\mathrm{i}$ is given below.



You can find the modulus using the formula from the study guide. Here, $a = 5$ and $b = -12$ and so:

$$|z| = \sqrt{(5)^2 + (-12)^2}$$

$$|z| = \sqrt{25 + 144}$$

$$|z| = \sqrt{169} = 13$$

You can see from your Argand diagram that $z = 5 - 12i$ is in quadrant 4. Remember from the study guide that the argument of a complex number *z* in quadrant 4 is given by the formula:

$$\text{Arg}(z) = \tan^{-1}\left(\frac{b}{a}\right)$$

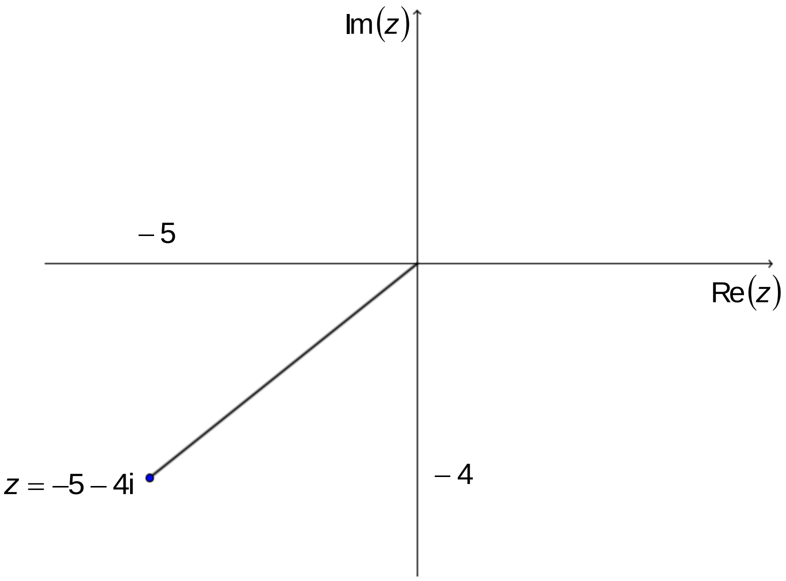
Here, $a = 5$ and $b = -12$ and so you can see that:

$$\text{Arg}(z) = \tan^{-1}\left(\frac{-12}{5}\right) = -1.176$$

So you can write $z = 5 - 12i$ in polar form as $z = 13e^{-1.176\,i}$.

iii) For $z = -5 - 4i$, $|z| = \sqrt{41}$, $\text{Arg}(z) = -2.467$ and so $z = \sqrt{41}e^{-2.467\,i}$ in polar form.

The Argand diagram for $z = -5 - 4i$ is given below.



You can find the modulus using the formula from the study guide. Here, $a = -5$ and $b = -4$ and so:

$$|z| = \sqrt{(-5)^2 + (-4)^2}$$

$$|z| = \sqrt{25+16}$$

$$|z| = \sqrt{41}$$

You can notice from your Argand diagram that $z = -5-4i$ is in quadrant 3. Remember from the study guide that the argument of a complex number z in quadrant 3 is given by the formula

$$\operatorname{Arg}(z) = -\pi + \tan^{-1}\left(\frac{b}{a}\right)$$

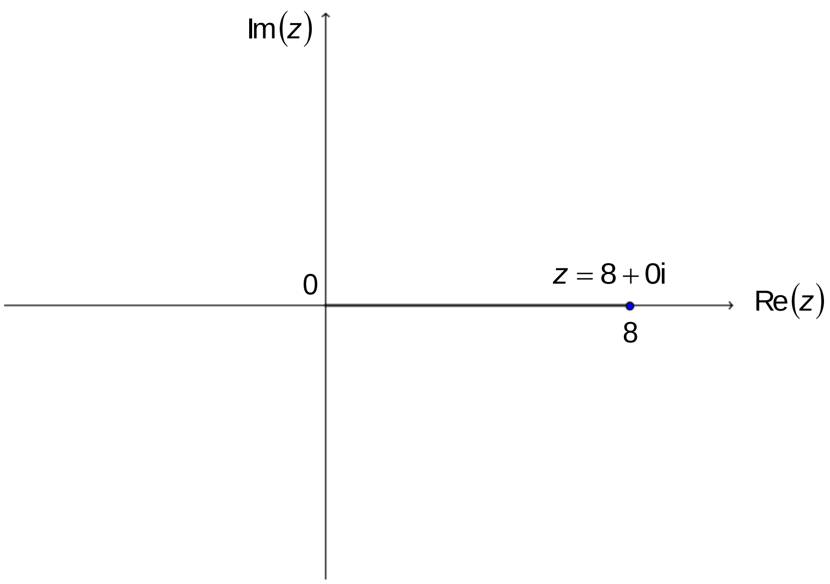
Here, $a = -5$ and $b = -4$ and so you can see that:

$$\operatorname{Arg}(z) = -\pi + \tan^{-1}\left(\frac{-4}{-5}\right) = -2.467$$

So you can write $z = -5-4i$ in polar form as $z = \sqrt{41}e^{-2.467\,i}$.

iv) For $z = 8+0i$, $|z| = 8$, $\operatorname{Arg}(z) = 0$ and so $z = 8e^{0i}$ in polar form.

The Argand diagram for $z = 8+0i$ is given below.



You can find the modulus using the formula from the study guide. Here, $a = 8$ and $b = 0$ and so:

$$|z| = \sqrt{(8)^2 + (0)^2}$$

$$|z| = \sqrt{64+0}$$

$$|z| = \sqrt{64} = 8$$

You can notice from your Argand diagram that $z = 8 + 0\text{i}$ is in quadrant 1. Remember from the study guide that the argument of a complex number z in quadrant 1 is given by the formula:

$$\text{Arg}(z) = \tan^{-1}\left(\frac{b}{a}\right).$$

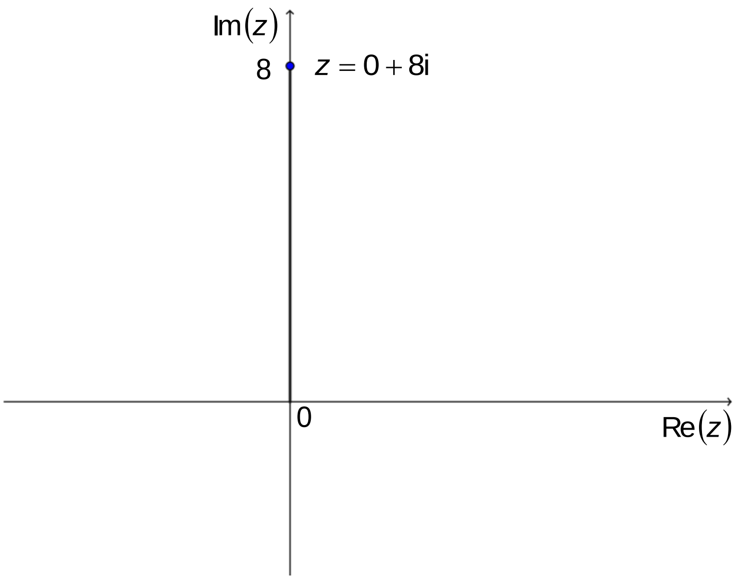
Here, $a = 8$, $b = 0$ and so you can see that:

$$\text{Arg}(z) = \tan^{-1}\left(\frac{0}{8}\right) = 0$$

So you can write $z = 8 + 0\text{i}$ in polar form as $z = 8e^{0\text{i}}$.

v) For $z = 0 + 8\text{i}$, $|z| = 8$, $\text{Arg}(z) = \pi/2$ and so $z = 8e^{\frac{\pi}{2}\text{i}}$ in polar form.

The Argand diagram for $z = 0 + 8\text{i}$ is given below.



You can find the modulus using the formula from the study guide. Here, $a = 0$ and $b = 8$ and so:

$$|z| = \sqrt{(0)^2 + (8)^2}$$

$$|z| = \sqrt{0+64}$$

$$|z| = \sqrt{64} = 8$$

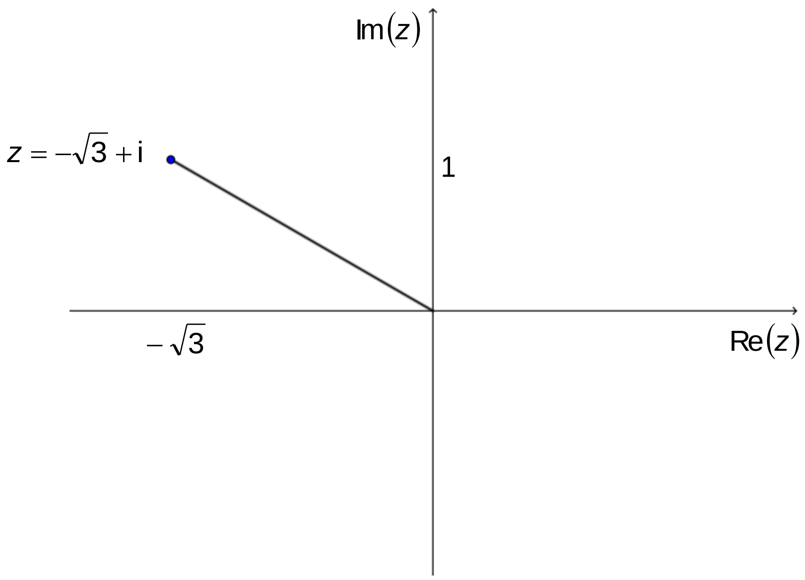
You can notice that $z = 0 + 8\mathrm{i}$ is a purely imaginary number with a positive *b*. Remember from the study guide that any such complex number *z* has argument $\pi/2$. So you can write:

$$\operatorname{Arg}(z) = \frac{\pi}{2}$$

So you can write $z = 0 + 8\mathrm{i}$ in polar form as $z = 8e^{\frac{\pi}{2}\mathrm{i}}$.

vi) For $z = -\sqrt{3} + \mathrm{i}$, $|z| = 2$, $\operatorname{Arg}(z) = \dfrac{5\pi}{6}$ and so $z = 2e^{\frac{5\pi}{6}\mathrm{i}}$ in polar form.

The Argand diagram for $z = -\sqrt{3} + \mathrm{i}$ is given below.



You can find the modulus using the formula from the study guide. Here, $a = -\sqrt{3}$ and $b = 1$ and so:

$$|z| = \sqrt{\left(-\sqrt{3}\right)^2 + (1)^2}$$

$$|z| = \sqrt{3+1}$$

$$|z| = \sqrt{4} = 2$$

You can notice from your Argand diagram that $z = -\sqrt{3} + \mathrm{i}$ is in quadrant 2.
Remember from the study guide that the argument of a complex number z in quadrant 2 is given by the formula:

$$\mathrm{Arg}(z) = \pi + \tan^{-1}\left(\frac{b}{a}\right)$$

Here, $a = -\sqrt{3}$, $b = 1$ and so you can see that:

$$\mathrm{Arg}(z) = \pi + \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

So you can write $z = -\sqrt{3} + \mathrm{i}$ in polar form as $z = 2e^{\frac{5\pi}{6}\mathrm{i}}$.

2. Remember from the study guide any complex number $z = re^{\mathrm{i}\theta}$ in polar form with modulus r and argument θ can be written as $z = a + b\mathrm{i}$, where:

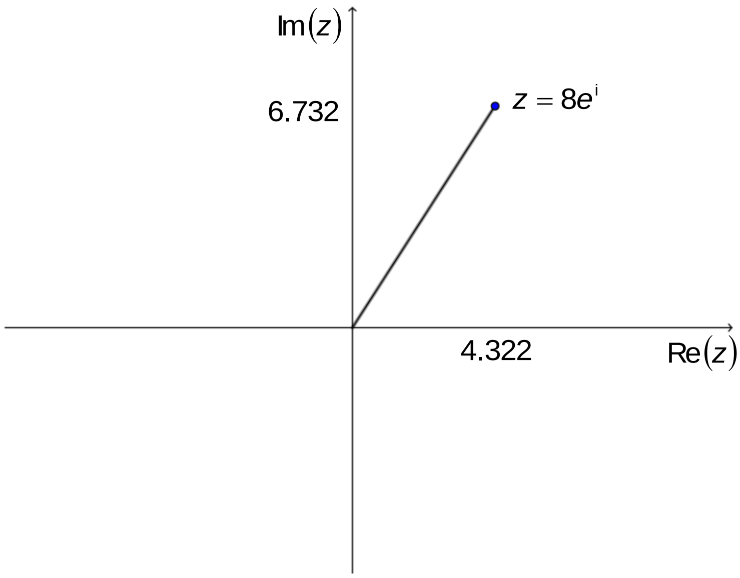
$$\mathrm{Re}(z) = a = r\cos\theta$$
$$\mathrm{Im}(z) = b = r\sin\theta.$$

i) For $z = 8e^{\mathrm{i}}$, the modulus is 8 and the argument is 1.
The Cartesian form of $z = 8e^{\mathrm{i}}$ is $z = 4.322 + 6.732\mathrm{i}$.

You can use the formulas above with $r = 8$ and $\theta = 1$ to get:

$$a = 8\cos 1 = 4.322$$
$$b = 8\sin 1 = 6.732$$

and so $z = 4.322 + 6.732\mathrm{i}$. See below for the Argand diagram of z.



ii) For $z = 8e^{\frac{\pi}{2}i}$, the modulus is 8 and the argument is $\pi/2$.

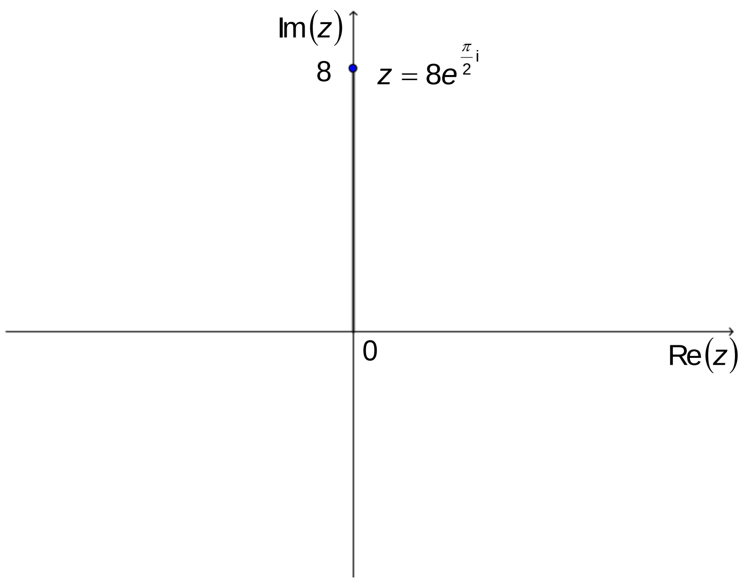
The Cartesian form of $z = 8e^{\frac{\pi}{2}i}$ is $z = 0 + 8i$.

You can use the formulas above with $r = 8$ and $\theta = \pi/2$ to get:

$$a = 8\cos\frac{\pi}{2} = 8 \cdot 0 = 0$$

$$b = 8\sin\frac{\pi}{2} = 8 \cdot 1 = 8$$

and so $z = 0 + 8i$. See below for the Argand diagram of *z*.



iii) For $z = 8e^{\pi i}$, the modulus is 8 and the argument is π.
The Cartesian form of $z = 8e^{\pi i}$ is $z = -8 + 0i$.

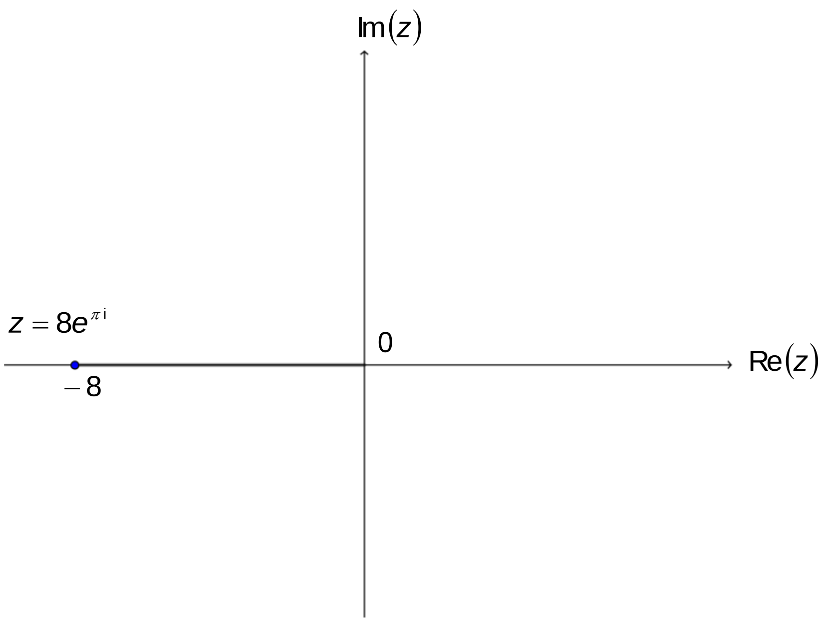
You can use the formulas above with $r = 8$ and $\theta = \pi$ to get:

$$a = 8\cos\pi = 8 \cdot (-1) = -8$$

And:

$$b = 8\sin\pi = 8 \cdot 0 = 0$$

and so $z = -8 + 0i$. See next page for the Argand diagram of *z*.



iv) For $z = \sqrt{8}e^{\frac{3\pi}{4}i}$, the modulus is 8 and the argument is $3\pi / 4$.

The Cartesian form of $z = \sqrt{8}e^{\frac{3\pi}{4}i}$ is $z = -2 + 2i$.

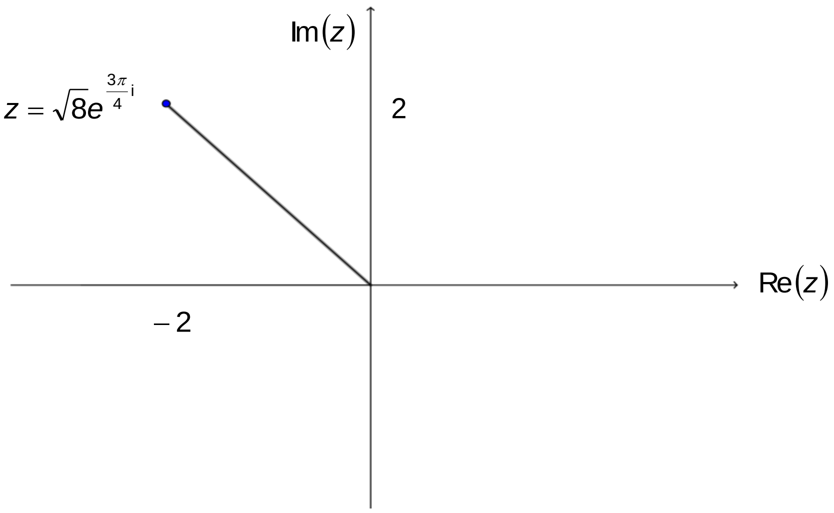
You can use the formulas above with $r = \sqrt{8}$ and $\theta = 3\pi/4$ to get:

$$a = \sqrt{8}\cos\frac{3\pi}{4}$$
$$a = \sqrt{8}\cdot\frac{-1}{\sqrt{2}} = -\sqrt{4} = -2$$

And:

$$b = \sqrt{8}\sin\frac{3\pi}{4} = 2$$

and so $z = -2 + 2i$. See below for the Argand diagram of *z*.

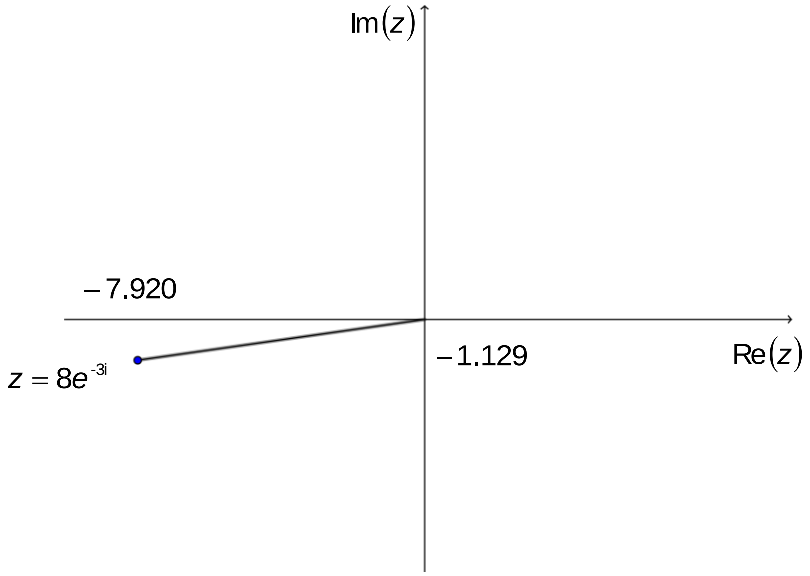


v) For $z = 8e^{-3i}$, the modulus is 8 and the argument is -3.
The Cartesian form of $z = 8e^{-3i}$ is $z = -7.920 - 1.129i$.

You can use the formulas above with $r = 8$ and $\theta = -3$ to get:

$$a = 8\cos(-3) = -7.920$$
$$b = 8\sin(-3) = -1.129$$

and so $z = -7.920 - 1.129i$. See below for the Argand diagram of *z*.



vi) For $z = \sqrt[3]{8}e^{-\frac{\pi}{2}i}$, the modulus is $\sqrt[3]{8} = 2$ and the argument is $-\pi/2$.

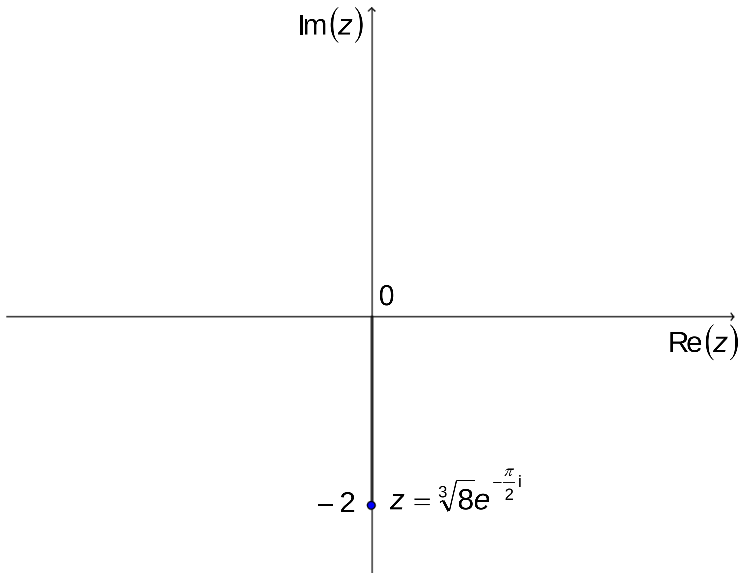
The Cartesian form of $z = \sqrt[3]{8}e^{-\frac{\pi}{2}i}$ is $z = 0 - 2i$.

You can use the formulas above with $r = 2$ and $\theta = -\pi/2$ to get:

$$a = 2\cos\frac{-\pi}{2} = 2 \cdot 0 = 0$$

$$b = 2\sin\frac{-\pi}{2} = 2 \cdot (-1) = -2$$

and so $z = 0 - 2i$. See the next page for the Argand diagram of *z*.



3. i) For $z = 6 + \sqrt{8}i$, the modulus $|z|$ is $\sqrt{44}$ and the argument $\text{Arg}(z)$ is 0.441.

You can find the modulus using the formula from the study guide. Here, $a = 6$, $b = \sqrt{8}$ and so:

$$|z| = \sqrt{(6)^2 + \left(\sqrt{8}\right)^2}$$
$$|z| = \sqrt{36 + 8}$$
$$|z| = \sqrt{44}$$

You can notice that $z = 6 + \sqrt{8}i$ is in quadrant 1. So using the formula for the argument of a complex number *z* in quadrant 1 gives:

$$\text{Arg}(z) = \tan^{-1}\left(\frac{\sqrt{8}}{6}\right) = 0.441$$

ii) For $z = -4 + \sqrt{28}i$, the modulus $|z|$ is $\sqrt{44}$ and the argument $\text{Arg}(z)$ is 2.218.

You can find the modulus using the formula from the study guide. Here, $a = -4$, $b = \sqrt{28}$ and so:

$$|z| = \sqrt{(-4)^2 + \left(\sqrt{28}\right)^2}$$
$$|z| = \sqrt{16 + 28}$$
$$|z| = \sqrt{44}$$

You can notice that $z = -4 + \sqrt{28}i$ is in quadrant 2. So using the formula for the argument of a complex number *z* in quadrant 2 gives:

$$\text{Arg}(z) = \pi + \tan^{-1}\left(\frac{\sqrt{28}}{-4}\right)$$
$$\text{Arg}(z) = 2.218.$$

iii) For $z = -4\sqrt{2} + 2\sqrt{14}i$, the modulus $|z|$ is $\sqrt{88}$ and the argument $\text{Arg}(z)$ is 2.218

You can find the modulus using the formula from the study guide. Here, $a = -4\sqrt{2}$, $b = 2\sqrt{14}$ and so:

$$|z| = \sqrt{\left(-4\sqrt{2}\right)^2 + \left(2\sqrt{14}\right)^2}$$
$$|z| = \sqrt{16 \cdot 2 + 4 \cdot 14}$$
$$|z| = \sqrt{32 + 56}$$
$$|z| = \sqrt{88}$$

You can notice that $z = -4\sqrt{2} + 2\sqrt{14}\,\mathrm{i}$ is in quadrant 2. So using the formula for the argument of a complex number *z* in quadrant 2 gives:

$$\mathrm{Arg}(z) = \pi + \tan^{-1}\left(\frac{2\sqrt{14}}{-4\sqrt{2}}\right)$$
$$\mathrm{Arg}(z) = \pi + \tan^{-1}\left(\frac{\sqrt{28}}{-4}\right)$$
$$\mathrm{Arg}(z) = 2.218\,.$$

iv) For $z = -8 - 2\sqrt{6}\mathrm{i}$, the modulus $|z|$ is $\sqrt{88}$ and the argument $\mathrm{Arg}(z)$ is -2.592.

You can find the modulus using the formula from the study guide. Here, $a = -8$, $b = -2\sqrt{6}$ and so:

$$|z| = \sqrt{(-8)^2 + \left(-2\sqrt{6}\right)^2}$$
$$|z| = \sqrt{64 + 4 \cdot 6}$$
$$|z| = \sqrt{64 + 24}$$
$$|z| = \sqrt{88}$$

You can notice that $z = -4\sqrt{2} - 2\sqrt{6}\,\mathrm{i}$ is in quadrant 3. So using the formula for the argument of a complex number *z* in quadrant 3 gives:

$$\mathrm{Arg}(z) = -\pi + \tan^{-1}\left(\frac{-2\sqrt{6}}{-8}\right)$$
$$\mathrm{Arg}(z) = -\pi + \tan^{-1}\left(\frac{\sqrt{6}}{4}\right)$$
$$\mathrm{Arg}(z) = -2.592\,.$$

v) For $z = -4 - \sqrt{6}\mathrm{i}$, the modulus $|z|$ is $\sqrt{22}$ and the argument $\mathrm{Arg}(z)$ is -2.592.

You can find the modulus using the formula from the study guide. Here, $a = -4$, $b = -\sqrt{6}$ and so:

$$|z| = \sqrt{(-4)^2 + \left(-\sqrt{6}\right)^2}$$

$$|z| = \sqrt{16 + 6}$$

$$|z| = \sqrt{22}$$

You can notice that $z = -4 - \sqrt{6}\mathrm{i}$ is in quadrant 3. So using the formula for the argument of a complex number *z* in quadrant 3 gives:

$$\mathrm{Arg}(z) = -\pi + \tan^{-1}\left(\frac{-\sqrt{6}}{-4}\right)$$

$$\mathrm{Arg}(z) = -2.592 \, .$$

vi) For $z = \sqrt{11} - \sqrt{11}\mathrm{i}$, the modulus $|z|$ is $\sqrt{22}$ and the argument $\mathrm{Arg}(z)$ is $-\pi/4$.

You can find the modulus using the formula from the study guide. Here, $a = \sqrt{11}$, $b = -\sqrt{11}$ and so:

$$|z| = \sqrt{\left(\sqrt{11}\right)^2 + \left(-\sqrt{11}\right)^2}$$

$$|z| = \sqrt{11 + 11}$$

$$|z| = \sqrt{22}$$

You can notice that $z = \sqrt{11} - \sqrt{11}\mathrm{i}$ is in quadrant 4. So using the formula for the argument of a complex number *z* in quadrant 4 gives:

$$\mathrm{Arg}(z) = \tan^{-1}\left(\frac{-\sqrt{11}}{\sqrt{11}}\right)$$

$$\mathrm{Arg}(z) = \tan^{-1}(-1)$$

$$\mathrm{Arg}(z) = -\frac{\pi}{4}$$

vii) For $z = \sqrt{22} - \sqrt{22}\,\mathrm{i}$, the modulus $|z|$ is $\sqrt{44}$ and the argument $\mathrm{Arg}(z)$ is $-\pi/4$.

You can find the modulus using the formula from the study guide. Here, $a = \sqrt{22}$, $b = -\sqrt{22}$ and so:

$$|z| = \sqrt{\left(\sqrt{22}\right)^2 + \left(-\sqrt{22}\right)^2}$$
$$|z| = \sqrt{22 + 22}$$
$$|z| = \sqrt{44}$$

You can notice that $z = \sqrt{22} - \sqrt{22}\,\mathrm{i}$ is in quadrant 4. So using the formula for the argument of a complex number *z* in quadrant 4 gives:

$$\mathrm{Arg}(z) = \tan^{-1}\left(\frac{-\sqrt{22}}{\sqrt{22}}\right)$$
$$\mathrm{Arg}(z) = \tan^{-1}(-1)$$
$$\mathrm{Arg}(z) = -\frac{\pi}{4}.$$

viii) You can see that $z = \sqrt{36} + 2\sqrt{2}\mathrm{i} = 6 + \sqrt{8}\mathrm{i}$. This number is the same as the number in i) and so the modulus $|z|$ is $\sqrt{44}$ and the argument $\mathrm{Arg}(z)$ is 0.441.

You can see that each consecutive pair of answers is linked. So i) and ii) have the same modulus, ii) and iii) have the same argument, and so on.

4. i) $z = e^{0\mathrm{i}}$ in Cartesian form is $z = 1 + 0\mathrm{i}$.

You can use the formulas above with $r = 1$ and $\theta = 0$ to get:

$$a = 1\cos 0 = 1 \cdot 1 = 1$$
$$b = 1\sin 0 = 1 \cdot 0 = 0$$

ii) $z = e^{\frac{\pi}{3}\mathrm{i}}$ in Cartesian form is $z = 1/2 + \sqrt{3}/2\,\mathrm{i}$.

You can use the formulas above with $r = 1$ and $\theta = \pi/3$ to get:

$$a = 1\cos\frac{\pi}{3} = 1 \cdot \frac{1}{2} = \frac{1}{2}$$
$$b = 1\sin\frac{\pi}{3} = 1 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

iii) $z = e^{\frac{2\pi}{3}\mathrm{i}}$ in Cartesian form is $z = -1/2 + \sqrt{3}/2\,\mathrm{i}$.

You can use the formulas above with $r = 1$ and $\theta = 2\pi/3$ to get:

$$a = 1\cos\frac{2\pi}{3} = 1\cdot\frac{-1}{2} = -\frac{1}{2}$$
$$b = 1\sin\frac{2\pi}{3} = 1\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

iv) $z = e^{\frac{3\pi}{3}\mathrm{i}} = e^{\pi\mathrm{i}}$ in Cartesian form is $z = -1 + 0\mathrm{i}$.

You can use the formulas above with $r = 1$ and $\theta = \pi$ to get:

$$a = 1\cos\pi = 1\cdot -1 = -1$$
$$b = 1\sin\pi = 1\cdot 0 = 0$$

(You may also recognise this as Euler's identity $e^{\pi\mathrm{i}} = -1$)

v) $z = e^{-\frac{\pi}{3}\mathrm{i}}$ in Cartesian form is $z = 1/2 - \left(\sqrt{3}/2\right)\mathrm{i}$.

You can use the formulas above with $r = 1$ and $\theta = -\pi/3$ to get:

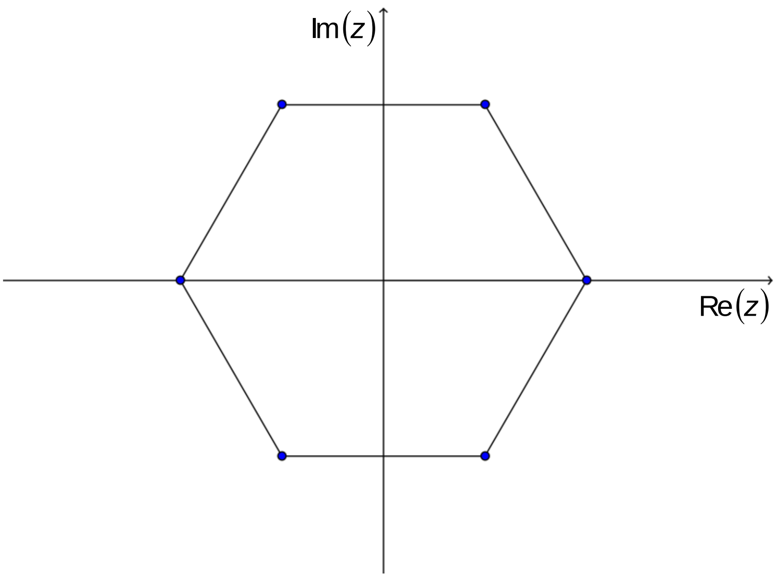
$$a = 1\cos\frac{-\pi}{3} = 1\cdot\frac{1}{2} = \frac{1}{2}$$
$$b = 1\sin\frac{-\pi}{3} = 1\cdot\frac{-\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$$

vi) $z = e^{-\frac{2\pi}{3}\mathrm{i}}$ in Cartesian form is $z = -1/2 - \left(\sqrt{3}/2\right)\mathrm{i}$.

You can use the formulas above with $r = 1$ and $\theta = -2\pi/3$ to get:

$$a = 1\cos\frac{-2\pi}{3} = 1\cdot\frac{-1}{2} = -\frac{1}{2}$$
$$b = 1\sin\frac{-2\pi}{3} = 1\cdot\frac{-\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$$

Look at the Argand diagram below. The shape here is a regular hexagon (all sides the same, all angles the same). This is not a coincidence. What you have just worked out is the 6^{th} roots of 1 in the complex plane.



You can work out the *n*th roots of 1 in the complex plane for any positive whole number *n*. It is a fact that the shape created by joining these complex numbers in the Argand diagram is a regular *n* sided shape.

vii) $z = e^{0i}$ in Cartesian form is $z = 1 + 0i$.

You can use the formulas above with $r = 1$ and $\theta = 0$ to get:

$$a = 1\cos 0 = 1 \cdot 1 = 1$$
$$b = 1\sin 0 = 1 \cdot 0 = 0$$

viii) $z = 2e^{\frac{\pi}{3}i}$ in Cartesian form is $z = 1 + \sqrt{3}\,i$.

You can use the formulas above with $r = 2$ and $\theta = \pi/3$ to get:

$$a = 2\cos\frac{\pi}{3} = 2 \cdot \frac{1}{2} = 1$$
$$b = 2\sin\frac{\pi}{3} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

ix) $z = 3e^{\frac{2\pi}{3}i}$ in Cartesian form is $z = -3/2 + 3\sqrt{3}/2\,i$.

You can use the formulas above with $r = 3$ and $\theta = 2\pi/3$ to get:

$$a = 3\cos\frac{2\pi}{3} = 3 \cdot \frac{-1}{2} = -\frac{3}{2}$$
$$b = 3\sin\frac{2\pi}{3} = 3 \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$$

x) $z = 4e^{\frac{3\pi}{3}i}$ in Cartesian form is $z = -4 + 0i$.

You can use the formulas above with $r = 4$ and $\theta = \pi$ to get:

$$a = 4\cos\pi = 4 \cdot 1 = 1$$
$$b = 4\sin\pi = 4 \cdot 0 = 0$$

xi) $z = 5e^{-\frac{2\pi}{3}i}$ in Cartesian form is $z = -5/2 - 5\sqrt{3}/2\,i$.

You can use the formulas above with $r = 5$ and $\theta = -2\pi/3$ to get:

$$a = 5\cos\frac{-2\pi}{3} = 5 \cdot \frac{-1}{2} = -\frac{5}{2}$$
$$b = 5\sin\frac{-2\pi}{3} = 5 \cdot \frac{-\sqrt{3}}{2} = -\frac{5\sqrt{3}}{2}$$

xii) $z = 6e^{-\frac{\pi}{3}i}$ in Cartesian form is $z = 3 - 3\sqrt{3}\,i$.

You can use the formulas above with $r = 6$ and $\theta = -\pi/3$ to get:

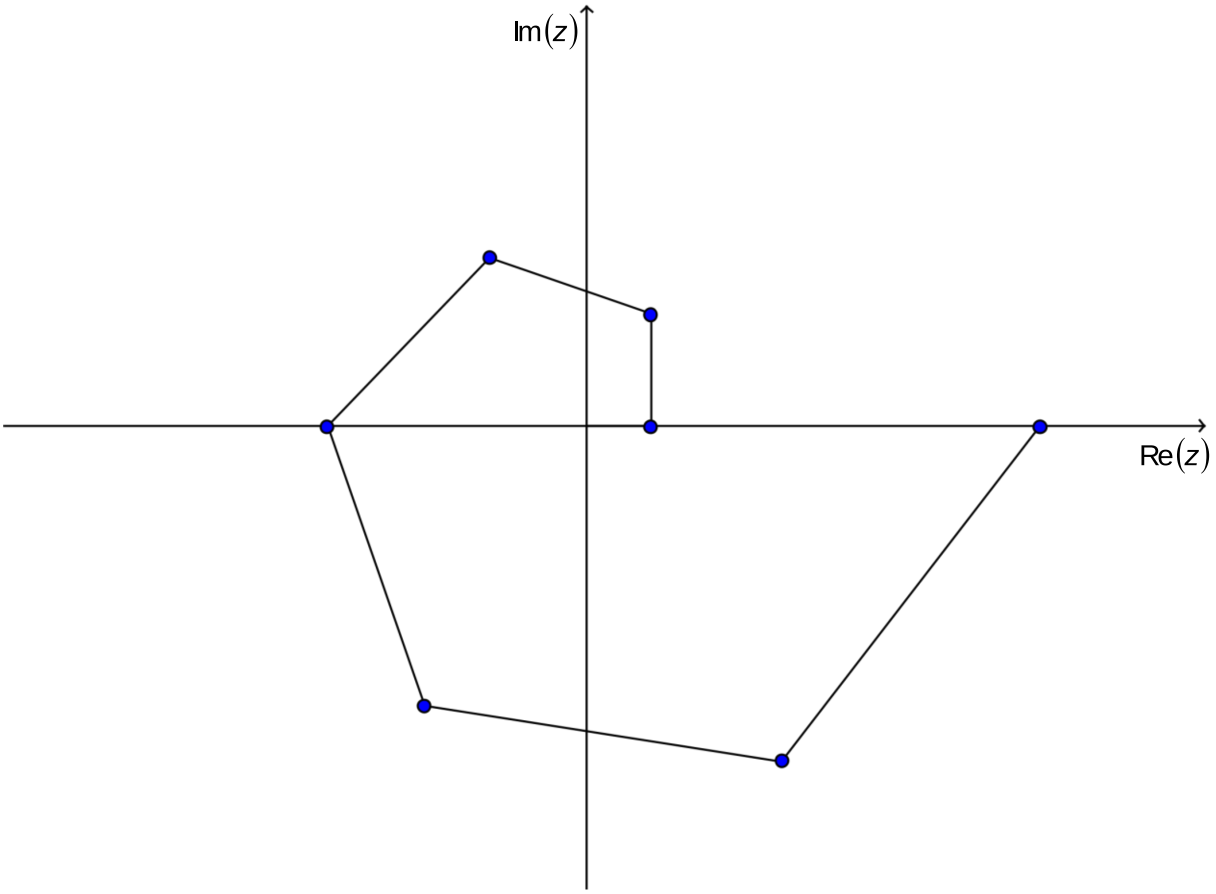
$$a = 6\cos\frac{-\pi}{3} = 6 \cdot \frac{1}{2} = 3$$
$$b = 6\sin\frac{-\pi}{3} = 6 \cdot \frac{-\sqrt{3}}{2} = -3\sqrt{3}$$

xiii) $z = 7e^{-\frac{3\pi}{3}i}$ in Cartesian form is $z = 7 + 0i$.

You can use the formulas above with $r = 7$ and $\theta = -\pi$ to get:

$$a = 7\cos(-\pi) = 7 \cdot 1 = 7$$
$$b = 7\sin(-\pi) = 7 \cdot 0 = 0$$

The Argand diagram is given below.


Im(z)
Re(z)