University of East Anglia
STUDENT SUPPORT SERVICE

Learning Enhancement Team

Steps into Complex Numbers

Basics of Complex Numbers

This guide introduces complex numbers and describes their definition and notation. It also shows you how to add, subtract, multiply and divide them and defines the complex conjugate.

Introduction

It is common to learn at school that you cannot have the square root of a negative number. This seems to make sense as the square of both positive and negative numbers are both positive numbers. For example:

both $3^2 = 9$ and $(-3)^2 = 9$

This this implies that $\sqrt{9}$ is both 3 and -3. Given this, the question "What number do you square to make -9?" seems impossible to answer as the square of both 3 and -3 is positive 9. You can write this question mathematically as:

$$x^2 = -9$$

and accept that this equation is impossible to solve. However mathematicians are always looking to solve equations that are seemingly impossible or difficult. The answer to how to solve such equations (and many others) is one of the most important, profound and controversial discoveries in the field of mathematics. It has its origin in the solution of certain types of cubic equations where the square root of negative numbers were evident in the method but conveniently cancelled out. Over time the idea of the square root of a negative number was accepted by mathematicians and was called an **imaginary number**. A new number was defined as:

$$\boxed{\mathrm{i} = \sqrt{-1}} \quad \text{or equivalently} \quad \boxed{\mathrm{i}^2 = -1}$$

So i is the square root of minus 1 and i squared is minus 1.† Any number which contains i is called a **complex number** by mathematicians. The existence of i allows you to write and work with the square root of a negative number without assuming that it is impossible to do so. This has proved extremely useful in pure mathematics and especially in physics where the solutions of many fundamental equations involve complex numbers.

† Although i is commonly used, in the study of electronics j is used instead as i represents current.

Example: Solve $x^2 = -9$.

Taking the square root of each side gives $x = \pm\sqrt{-9}$. The answer is the square root of a negative number. As $-9 = (-1) \times 9$ you can use the laws of indices to separate the root into $\sqrt{-9} = \sqrt{-1} \times \sqrt{9}$. As $\sqrt{9} = \pm 3$ and, as you have just seen, $\sqrt{-1} = \mathrm{i}$ the answer becomes:

$$x = \pm\sqrt{-9} = \pm 3\mathrm{i}$$

This example illustrates an important point: there is no need to define an infinite amount of new numbers to cope with the square root of a negative number, you only need i. This is because, for any negative number $-n$:

$$\sqrt{-n} = \sqrt{n \times (-1)} = \sqrt{n} \times \sqrt{-1} = \sqrt{n}\,\mathrm{i}$$

How do complex numbers relate to other numbers?

There are various ways to write a complex number, this guide will concentrate on the **Cartesian form** of a complex number. It is very common for a complex number to be called *z* and in Cartesian form *z* is written:

$z = a + b\mathrm{i}$	Cartesian form of a complex number

Where *a* and *b* are **real numbers** (see study guide: *Different Kinds of Numbers* for more details about real numbers) and have special names:

> *a* is the **real part** of *z*, often written $\mathrm{Re}(z)$. So $\mathrm{Re}(z) = a$
> *b* is the **imaginary part** of *z*, often written $\mathrm{Im}(z)$. So $\mathrm{Im}(z) = b$

it should be emphasised that the imaginary part of a complex number is the real number that is multiplied by i, and does not contain i.

> When $a = 0$, $z = b\mathrm{i}$ and is called an **imaginary number**
> When $b = 0$, $z = a$ and is a **real number**.

Example: What are the real and imaginary parts of the following numbers?
(a) $z = 5 + 3\mathrm{i}$ (b) $z = -2 - \mathrm{i}$ (c) $z = -6\mathrm{i}$ (d) $z = 3.4$

(a) $\mathrm{Re}(z) = 5$ and $\mathrm{Im}(z) = 3$.

(b) $\mathrm{Re}(z) = -2$ and $\mathrm{Im}(z) = -1$. Remember to include the sign in your answer.

(c) This is an imaginary number so $\mathrm{Re}(z) = 0$ and $\mathrm{Im}(z) = -6$.

(d) This is a real number so $\mathrm{Re}(z) = 3.4$ and $\mathrm{Im}(z) = 0$.

Sometimes a complex number is not expressed exactly as $z = a + b\text{i}$ and it is useful to manipulate the number to help you determine the real and imaginary parts.

Example: Solve $x^2 + x + 1 = 0$.

This is a quadratic equation and you can use the quadratic formula to solve it (see study guide: *Solving Quadratic Equations using the Quadratic Formula* for details). This gives:

$$x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}\,\text{i}}{2}$$

So the two solutions are $x_1 = \dfrac{-1 + \sqrt{3}\,\text{i}}{2}$ and $x_2 = \dfrac{-1 - \sqrt{3}\,\text{i}}{2}$.

However, as the solutions are complex numbers it is useful to separate the real and imaginary parts to express them in Cartesian form as $z = a + b\text{i}$:

$x_1 = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\text{i}$ you can see that $\text{Re}(x_1) = -\dfrac{1}{2}$ and $\text{Im}(x_1) = \dfrac{\sqrt{3}}{2}$.

$x_2 = -\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}\text{i}$ you can see that $\text{Re}(x_2) = -\dfrac{1}{2}$ and $\text{Im}(x_2) = -\dfrac{\sqrt{3}}{2}$.

When the **discriminant** of the quadratic formula is **negative** you will always get complex numbers as solutions for *x*.

The complex conjugate

Every complex number has another complex number associated with it called its **complex conjugate**. The complex conjugate of a complex number *z* is written as $\bar{z}$ (or often as *z**) and is found by changing the sign of the imaginary part of a complex number:

If $z = a + b\text{i}$ then $\bar{z} = a - b\text{i}$

Example: What are the complex conjugates of the following numbers?

(a) $z = 5 + 3\text{i}$ (b) $z = -2 - \text{i}$ (c) $z = -6\text{i}$ (d) $z = 3.4$

(a) If $z = 5 + 3\text{i}$ then $\bar{z} = 5 - 3\text{i}$.

(b) If $z = -2 - \text{i}$ then $\bar{z} = -2 + \text{i}$.

(c) If $z = -6\text{i}$ then $\bar{z} = 6\text{i}$.

(d) If $z = 3.4$ then $\bar{z} = 3.4$, the complex conjugate of a real number is the number itself.

You should also notice that the solutions to the quadratic equation in the example in the previous section are also complex conjugates of each other. In fact, when the solutions to a quadratic equation are two complex numbers they are always complex conjugates of each other.

Adding and subtracting complex numbers

Adding and subtracting complex numbers is relatively easy, you simply add (or subtract as necessary) the real parts of the numbers to find the real part of the answer and add (or subtract) the imaginary parts of the numbers to find the imaginary part of the answer.

Example: If you have two complex numbers $z_1 = 5 + 3\mathrm{i}$ and $z_2 = -2 - \mathrm{i}$ what is (a) $z_1 + z_2$ and (b) $z_1 - z_2$?

(a) You can write $z_1 + z_2 = (5 + 3\mathrm{i}) + (-2 - \mathrm{i})$.

Adding the real parts gives the real part of the answer: $5 + (-2) = 3$

Adding the imaginary parts gives the imaginary part of the answer: $3 + (-1) = 2$

So: $z_1 + z_2 = 3 + 2\mathrm{i}$

(b) You can write $z_1 - z_2 = (5 + 3\mathrm{i}) - (-2 - \mathrm{i})$.

Subtracting the real parts gives the real part of the answer: $5 - (-2) = 7$

Subtracting the imaginary parts gives the imaginary part of the answer: $3 - (-1) = 4$

So: $z_1 - z_2 = 7 + 4\mathrm{i}$.

Multiplying complex numbers

You should now know that a complex number has two parts, real and imaginary, and this complicates their multiplication. However it is the same as opening brackets with two terms in each and in order to multiply complex numbers successfully you should employ a method which opens such brackets. The study guide: *Opening Brackets* illustrates a grid method.

Example: If you have two complex numbers $z_1 = 5 + 3\mathrm{i}$ and $z_2 = -2 - \mathrm{i}$ what is $z_1 z_2$?

You can write $z_1 z_2 = (5 + 3\mathrm{i})(-2 - \mathrm{i})$ which reveals the reason why you need to open brackets to find the answer. So:

$$z_1 z_2 = (5 + 3\mathrm{i})(-2 - \mathrm{i}) = -10 - 6\mathrm{i} - 5\mathrm{i} - 3\mathrm{i}^2$$

It is important that you remember that $\mathrm{i}^2 = -1$ and so the final term $-3\mathrm{i}^2 = (-3) \times (-1) = +3$.

You can use this to help **collect** together the real and imaginary parts of the answer to give:

$$z_1 z_2 = (5+3\mathrm{i})(-2-\mathrm{i}) = (-10+3)-(6+5)\mathrm{i} = -7-11\mathrm{i}$$

So the answer is a complex number expressed in Cartesian form with a real part of -7 and an imaginary part of -11.

When you multiply a complex number by its complex conjugate you get a result that proves to be both important and useful.

Example: Given that $z = 5+3\mathrm{i}$, what is $z\bar{z}$?

You can write $z\bar{z} = (5+3\mathrm{i})(5-3\mathrm{i})$ and so:

$$z\bar{z} = (5+3\mathrm{i})(5-3\mathrm{i}) = +25+15\mathrm{i}-15\mathrm{i}-9\mathrm{i}^2 = 25+9 = 36$$

Cancelling out the second and third terms and using $\mathrm{i}^2 = -1$. Multiplying a complex number by its complex conjugate is important and useful because the imaginary part cancels out and the answer is a purely real number.

Multiplying a complex number by its complex conjugate gives a purely real number.

Using the general complex number $z = a + b\mathrm{i}$:

$$z\bar{z} = (a+b\mathrm{i})(a-b\mathrm{i}) = a^2 + b^2$$

This is **the sum of two squares** which is similar to the difference of two squares (a common piece of mathematics discussed in the study guide: *Factorising Quadratic Expressions*). The boxed mathematics above illustrates how you can factorise the sum of two squares.

Dividing complex numbers

Dividing complex numbers is more challenging than their addition, subtraction or multiplication. Let's look at an example to help understand the process.

Example: If you have two complex numbers $z_1 = 5+\mathrm{i}$ and $z_2 = -2-\mathrm{i}$ what is $\frac{z_1}{z_2}$?

You can write $\frac{z_1}{z_2} = \frac{5+\mathrm{i}}{-2-\mathrm{i}}$.

The difficulty here is that a complex number has a real part and an imaginary part but in this piece of mathematics it is difficult to see what they are. The method uses the fact discussed in the previous section that multiplying a complex number by its complex conjugate gives a real number. If you had a real number as the denominator of the division you can split the complex number up into real and imaginary parts. So you need to multiply the denominator by its complex conjugate $-2+\mathrm{i}$ to get a real number. Remember that you must also multiply the numerator by the same thing to keep the fractions equivalent so:

$$\frac{z_1}{z_2}=\frac{5+\mathrm{i}}{-2-\mathrm{i}}=\frac{(5+\mathrm{i})(-2+\mathrm{i})}{(-2-\mathrm{i})(-2+\mathrm{i})}=\frac{-10+5\mathrm{i}-2\mathrm{i}+\mathrm{i}^2}{4-2\mathrm{i}+2\mathrm{i}-\mathrm{i}^2}=\frac{-11+3\mathrm{i}}{5}$$

This piece of mathematics makes use of many of the ideas discussed in this guide such as expanding brackets, collecting together real and imaginary parts of a complex number, complex conjugates and $\mathrm{i}^2=-1$. You are still not finished however, to give the answer in Cartesian form you must separate the real and imaginary parts of the complex number:

$$\frac{z_1}{z_2}=\frac{-11+3\mathrm{i}}{5}=-\frac{11}{5}+\frac{3}{5}\mathrm{i}$$

Which shows that the real part of the answer is $-\frac{11}{5}$ and the imaginary part is $\frac{3}{5}$.

Want to know more?

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