Model answers: Basics of Complex Numbers

Basics of Complex Numbers study guide

Remember from the study guide that:

- the Cartesian form of a complex number is given by $z = a + bi$:
- For a complex number $z = a + bi$, the real part is a and you can write $\text{Re}(z) = a$.
  The imaginary part of z is b and you can write $\text{Im}(z) = b$.
  Remember that the imaginary part is the number multiplied by i, and does not include i.
- For a complex number $z = a + bi$, its complex conjugate is $\bar{z} = a - bi$

1. i) $\sqrt{-9} = \sqrt{9} \times \sqrt{-1} = 3\text{i}$, so $\sqrt{-9}$ in Cartesian form is $0 + 3\text{i}$.

   You can see from this that $\text{Re}\left(\sqrt{-9}\right) = 0$ and $\text{Im}\left(\sqrt{-9}\right) = 3$.

   The complex conjugate of $\sqrt{-9} = 0 + 3\text{i}$ is $0 - 3\text{i}$.

   ii) $\sqrt{-36} = \sqrt{36} \times \sqrt{-1} = 6\text{i}$, so $\sqrt{-36}$ in Cartesian form is $0 + 6\text{i}$.

   You can see from this that $\text{Re}\left(\sqrt{-36}\right) = 0$ and $\text{Im}\left(\sqrt{-36}\right) = 6$.

   The complex conjugate of $\sqrt{-6} = 0 + 6\text{i}$ is $0 - 6\text{i}$.

   iii) You can use your answer to ii) to see that $\sqrt{9} + \sqrt{-36} = 3 + 6\text{i}$, so $\sqrt{9} + \sqrt{-36}$ in Cartesian form is $3 + 6\text{i}$.

   You can see from this that $\text{Re}\left(\sqrt{9} + \sqrt{-36}\right) = 3$ and $\text{Im}\left(\sqrt{9} + \sqrt{-36}\right) = 6$.

   The complex conjugate of $\sqrt{9} + \sqrt{-36} = 3 + 6\text{i}$ is $3 - 6\text{i}$.

iv) You can use your answer to i) to see that $\sqrt{-9}-\sqrt{36}=3i-6$, so $\sqrt{-9}-\sqrt{36}$ in Cartesian form is $-6+3i$.

You can see from this that $\text{Re}\left(\sqrt{-9}-\sqrt{36}\right)=-6$ and $\text{Im}\left(\sqrt{-9}-\sqrt{36}\right)=3$.

The complex conjugate of $\sqrt{-9}-\sqrt{36}=-6+3i$ is $-6-3i$.

v) $\sqrt{9}+\sqrt{36}=3+6=9$, so $\sqrt{9}+\sqrt{36}$ in Cartesian form is $9+0i$.

You can see from this that $\text{Re}\left(\sqrt{9}+\sqrt{36}\right)=9$ and $\text{Im}\left(\sqrt{9}+\sqrt{36}\right)=0$.

The complex conjugate of $\sqrt{9}+\sqrt{36}=9+0i$ is $9-0i=9$.

vi) You can see that $-(9)^2=-81$. Using this, $\sqrt{-(9)^2}=\sqrt{81}\times\sqrt{-1}=9i$, and so $\sqrt{-(9)^2}$ in Cartesian form is $0+9i$.

You can see from this that $\text{Re}\left(\sqrt{-(9)^2}\right)=0$ and $\text{Im}\left(\sqrt{-(9)^2}\right)=9$.

The complex conjugate of $\sqrt{-(9)^2}=0+9i$ is $0-9i$.

vii) You can see that $(-9)^2=81$. Using this, $\sqrt{(-9)^2}=\sqrt{81}=9$, and so $\sqrt{(-9)^2}$ in Cartesian form is $9+0i$. Note that this is not the same as vi).

You can see from this that $\text{Re}\left(\sqrt{(-9)^2}\right)=0$ and $\text{Im}\left(\sqrt{(-9)^2}\right)=0$.

The complex conjugate of $\sqrt{(-9)^2}=9+0i$ is $9-0i=9$.

viii) Remember that $i^2=-1$, so $6i^2=-6$. Using this, $3i+6i^2=3i-6$ and so $3i+6i^2$ in Cartesian form is $-6+3i$.

You can see from this that $\text{Re}\left(3i+6i^2\right)=-6$ and $\text{Im}\left(3i+6i^2\right)=3$.

The complex conjugate of $3i+6i^2=-6+3i$ is $-6-3i$.

ix) You can use your answer to i) to see that $3\mathrm{i}-\sqrt{-9}=3\mathrm{i}-3\mathrm{i}=0$, so $3\mathrm{i}-\sqrt{-9}$ in Cartesian form is $0+0\mathrm{i}$.

You can see from this that $\mathrm{Re}\left(3\mathrm{i}-\sqrt{-9}\right)=0$ and $\mathrm{Im}\left(3\mathrm{i}-\sqrt{-9}\right)=0$.

The complex conjugate of $3\mathrm{i}-\sqrt{-9}=0+0\mathrm{i}$ is $0-0\mathrm{i}$.

x) Remember that $\mathrm{i}^2=-1$, so $-3\mathrm{i}^2=3$. Using this, $9+6\mathrm{i}-3\mathrm{i}^2=9+6\mathrm{i}+3$ and so $9+6\mathrm{i}-3\mathrm{i}^2$ in Cartesian form is $12+6\mathrm{i}$.

You can see from this that $\mathrm{Re}\left(9+6\mathrm{i}-3\mathrm{i}^2\right)=12$ and $\mathrm{Im}\left(9+6\mathrm{i}-3\mathrm{i}^2\right)=6$.

The complex conjugate of $9+6\mathrm{i}-3\mathrm{i}^2=12+6\mathrm{i}$ is $12-6\mathrm{i}$.

xi) Remember that $\mathrm{i}^2=-1$, so $36\mathrm{i}^2=-36$. Using this, $3\mathrm{i}-\sqrt{36\mathrm{i}^2}=3\mathrm{i}-\sqrt{-36}=3\mathrm{i}-6\mathrm{i}=-3\mathrm{i}$ and so $3\mathrm{i}-\sqrt{36\mathrm{i}^2}$ in Cartesian form is $0-6\mathrm{i}$.

You can see from this that $\mathrm{Re}(0-6\mathrm{i})=0$ and $\mathrm{Im}(0-6\mathrm{i})=-6$.

The complex conjugate of $0-6\mathrm{i}$ is $0+6\mathrm{i}$.

xii) Remember that $\mathrm{i}^2=-1$, so $9\mathrm{i}^2=-9$ and $(-9\mathrm{i})^2=81\mathrm{i}^2=-81$. Using your answers to parts i) and vi), $\sqrt{9\mathrm{i}^2}\sqrt{(-9\mathrm{i})^2}=\sqrt{-9}\sqrt{-81}=(3\mathrm{i})(9\mathrm{i})=27\mathrm{i}^2=-27$. and so $\sqrt{9\mathrm{i}^2}\sqrt{(-9\mathrm{i})^2}$ in Cartesian form is $-27+0\mathrm{i}$.

You can see from this that $\mathrm{Re}\left(\sqrt{9\mathrm{i}^2}\sqrt{(-9\mathrm{i})^2}\right)=-27$ and $\mathrm{Im}\left(\sqrt{9\mathrm{i}^2}\sqrt{(-9\mathrm{i})^2}\right)=0$.

The complex conjugate of $\sqrt{9\mathrm{i}^2}\sqrt{(-9\mathrm{i})^2}=-27+0\mathrm{i}$ is $-27-0\mathrm{i}=-27$.

2. Remember that for a quadratic equation in the form $px^2+qx+r=0$ (with $p\neq 0$), the roots x of the quadratic equation are given by the formula:

$$x=\frac{-q\pm\sqrt{q^2-4pr}}{2p}$$

(For more information see study guide: *Solving Quadratic Equations using the Quadratic Formula*)

i) $a = -1 + \mathrm{i}$ or $a = -1 - \mathrm{i}$.

For $a^2 + 2a + 2 = 0$, you can see here that $p = 1$, $q = 2$ and $r = 2$. Putting these into the quadratic formula above, you see that:

$$a = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(2)}}{2(1)} = \frac{-2 \pm \sqrt{-4}}{2}.$$

You can use a similar argument to 1i) to see that $\sqrt{-4} = \sqrt{4} \times \sqrt{-1} = 2\mathrm{i}$ and so:

$$a = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2\mathrm{i}}{2} = -1 \pm \mathrm{i}$$

and so *a* is either $-1 + \mathrm{i}$ or it is $-1 - \mathrm{i}$.

ii) $b = 13\mathrm{i}$ or $b = -13\mathrm{i}$.

For $b^2 + 169 = 0$, you can see here that $p = 1$, $q = 0$ and $r = 169$. Putting these into the quadratic formula above, you see that:

$$b = \frac{0 \pm \sqrt{(0)^2 - 4(1)(169)}}{2(1)} = \frac{\pm \sqrt{-676}}{2}.$$

You can use a similar argument to 1.i) to see that $\sqrt{-676} = \sqrt{676} \times \sqrt{-1} = 26\mathrm{i}$ and so:

$$b = \frac{\pm \sqrt{-676}}{2} = \frac{\pm 26\mathrm{i}}{2} = \pm 13\mathrm{i}$$

and so *b* is either $13\mathrm{i}$ or it is $-13\mathrm{i}$.

You could also subtract 169 from each side and then take the square root to find the same answer.

iii) $c = -2$ is a double root of this equation.

For $c^2 + 4c + 4 = 0$, you can see here that $p = 1$, $q = 4$ and $r = 4$. Putting these into the quadratic formula above, you see that:

$$c = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(4)}}{2(1)} = \frac{-4 \pm \sqrt{0}}{2}.$$

And so:

$$c = \frac{-4 \pm 0}{2} = \frac{-4}{2} = -2$$

and so c is -2. Remember that quadratic equations always have two roots, and so you can say that both roots of this equation are equal to -2.

iv) $d = -2 + \mathrm{i}$ or $d = -2 - \mathrm{i}$.

For $d^2 + 4d + 5 = 0$, you can see here that $p = 1$, $q = 4$ and $r = 5$. Putting these into the quadratic formula above, you see that:

$$d = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(5)}}{2(1)} = \frac{-4 \pm \sqrt{-4}}{2}.$$

You can use a similar argument to 1.i) to see that $\sqrt{-4} = \sqrt{4} \times \sqrt{-1} = 2\mathrm{i}$ and so:

$$d = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2\mathrm{i}}{2} = -2 \pm \mathrm{i}$$

and so d is either $-2 + \mathrm{i}$ or it is $-2 - \mathrm{i}$.

v) $e = -3/2 - \sqrt{3}/2\mathrm{i}$ or $e = -3/2 + \sqrt{3}/2\mathrm{i}$.

For $(e\mathrm{i})^2 - 3e - 3 = 0$, you can see here that it is not quite in the form $px^2 + qx + r = 0$.

You can however see that

$$(e\mathrm{i})^2 = e^2\mathrm{i}^2 = -e^2$$

and so

$$(e\mathrm{i})^2 - 3e - 3 = -e^2 - 3e - 3 = 0.$$

which is in the form $px^2 + qx + r = 0$. So you have that for this quadratic, $p = -1$, $q = -3$ and $r = -3$. Putting these into the quadratic formula above:

$$e = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(-1)(-3)}}{2(-1)} = \frac{3 \pm \sqrt{9 - 12}}{-2} = \frac{3 \pm \sqrt{-3}}{-2}.$$

You can use a similar argument to 1.i) to see that $\sqrt{-3} = \sqrt{3} \times \sqrt{-1} = \sqrt{3}\,\mathrm{i}$ and so:

$$e = \frac{3 \pm \sqrt{-3}}{-2} = \frac{3 \pm \sqrt{3}\,\mathrm{i}}{-2} = -\frac{3}{2} \pm -\frac{\sqrt{3}}{2}\mathrm{i}$$

and so e is either $-3/2 - \sqrt{3}/2\mathrm{i}$ or it is $-3/2 + \sqrt{3}/2\mathrm{i}$.

vi) $f = -2$ is a double root of this equation..

For $-(f\mathrm{i})^2 - 4\mathrm{i}^2 f - (2\mathrm{i})^2 = 0$, you can see here that it is not quite in the form $px^2 + qx + r = 0$.

You can however see that

$$-(f\mathrm{i})^2 = -\left(f^2\mathrm{i}^2\right) = -\left(-f^2\right) = f^2$$

and:

$$4\mathrm{i}^2 f = \left(-4f\right) = -4f$$

and:

$$(2\mathrm{i})^2 = \left(4\mathrm{i}^2\right) = -4.$$

Therefore the quadratic becomes

$$-(f\mathrm{i})^2 - 4\mathrm{i}^2 f - (2\mathrm{i})^2 = f^2 - \left(-4f\right) - \left(-4\right) = f^2 + 4f + 4 = 0$$

which is in the form $px^2 + qx + r = 0$. So you have that for this quadratic, $p = 1$, $q = 4$ and $r = 4$. You can notice that this is exactly the same equation as found in part iii), so you can use your answer for part iii) and say that $f = -2$ is a double root of this equation.

3. i) $z_1 + z_3 = \left(4 - 3\mathrm{i}\right) + \left(1 + \mathrm{i}\right) = 4 + 1 - 3\mathrm{i} + \mathrm{i} = 5 - 2\mathrm{i}.$

You can see from this that $\mathrm{Re}\left(z_1 + z_3\right) = 5$ and $\mathrm{Im}\left(z_1 + z_3\right) = -2$.

ii) $z_2 - z_1 = \left(-2 + 2\mathrm{i}\right) - \left(4 - 3\mathrm{i}\right) = -2 - 4 + 2\mathrm{i} + 3\mathrm{i} = -6 + 5\mathrm{i}.$

You can see from this that $\mathrm{Re}\left(z_2 - z_1\right) = -6$ and $\mathrm{Im}\left(z_2 - z_1\right) = 5$.

iii) $z_3 z_2 = -4$.

You can see from this that $\text{Re}(z_3 z_2) = -4$ and $\text{Im}(z_3 z_2) = 0$.

You have that

$$z_3 z_2 = (1+i)(-2+2i)$$

Expanding the brackets gives:

$$z_3 z_2 = (1+i)(-2+2i) = -2+2i-2i+2i^2$$

You can simplify using the fact that $2i^2 = -2$ to get:

$$z_3 z_2 = -2+2i-2i+2i^2 = -2-2 = -4$$

iv) $\dfrac{z_1}{z_2} = -\dfrac{7}{4} - \dfrac{1}{4}i$.

You can see from this that $\text{Re}(z_1 / z_2) = -7/4$ and $\text{Im}(z_1 / z_2) = -1/4$

You have that

$$\frac{z_1}{z_2} = \frac{(4-3i)}{(-2+2i)}$$

Remember from the study guide that when you are dividing one complex number by another, you need to make the denominator real. You can do this by multiplying both top and bottom of the division to be done by the complex conjugate of the denominator. You can see that $\bar{z}_2 = -2-2i$ and so:

$$\frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{z_2 \bar{z}_2} = \frac{(4-3i)(-2-2i)}{(-2+2i)(-2-2i)}$$

Using a similar argument to part iii), you can expand the brackets on both the numerator and the denominator and simplify to get:

$$\frac{z_1}{z_2} = \frac{(4-3i)(-2-2i)}{(-2+2i)(-2-2i)} = \frac{-8-8i+6i+6i^2}{4+4i-4i-4i^2} = \frac{-8+6(-1)-2i}{4-(-4)} = \frac{-14-2i}{8}$$

This expression still needs to be written in Cartesian form. You can see that:

$$\frac{z_1}{z_2} = \frac{-14-2i}{8} = -\frac{14}{8} - \frac{2i}{8} = -\frac{7}{4} - \frac{1}{4}i$$

To work out v)-viii):

the complex conjugate of $z_1 = 4 - 3i$ is $\bar{z}_1 = 4 + 3i$:
the complex conjugate of $z_2 = -2 + 2i$ is $\bar{z}_2 = -2 - 2i$:
the complex conjugate of $z_3 = 1 + i$ is $\bar{z}_3 = 1 - i$.

v) $\bar{z}_1 + \bar{z}_3 = (4 + 3i) + (1 - i) = 4 + 1 + 3i - i = 5 + 2i.$

You can see from this that $\mathrm{Re}(\bar{z}_1 + \bar{z}_3) = 5$ and $\mathrm{Im}(\bar{z}_1 + \bar{z}_3) = 2$.

vi) $\bar{z}_2 - \bar{z}_1 = (-2 - 2i) - (4 + 3i) = -2 - 4 - 2i - 3i = -6 - 5i.$

You can see from this that $\mathrm{Re}(\bar{z}_2 - \bar{z}_1) = -6$ and $\mathrm{Im}(\bar{z}_2 - \bar{z}_1) = -5$.

vii) $\bar{z}_3 \bar{z}_2 = -4$.

You can see from this that $\mathrm{Re}(\bar{z}_3 \bar{z}_2) = -4$ and $\mathrm{Im}(\bar{z}_3 \bar{z}_2) = 0$.

You have that

$$\bar{z}_3 \bar{z}_2 = (1 - i)(-2 - 2i)$$

Expanding the brackets gives:

$$\bar{z}_3 \bar{z}_2 = (1 - i)(-2 - 2i) = -2 - 2i + 2i + 2i^2$$

You can simplify using the fact that $2i^2 = -2$ to get:

$$\bar{z}_3 \bar{z}_2 = -2 - 2i + 2i + 2i^2 = -2 - 2 = -4.$$

viii) $\dfrac{\bar{z}_1}{\bar{z}_2} = -\dfrac{7}{4} + \dfrac{1}{4}i.$

You can see from this that $\mathrm{Re}(\bar{z}_1 / \bar{z}_2) = -7/4$ and $\mathrm{Im}(\bar{z}_1 / \bar{z}_2) = 1/4$

You have that

$$\frac{\bar{z}_1}{\bar{z}_2} = \frac{(4 + 3i)}{(-2 - 2i)}$$

Remember from the study guide that when you are dividing one complex number by another, you need to make the denominator real. You can do this by multiplying both top and bottom of the division to be done by the complex conjugate of the denominator. You can see that $z_2 = -2 + 2\text{i}$ and so:

$$\frac{\bar{z}_1}{\bar{z}_2} = \frac{\bar{z}_1 z_2}{\bar{z}_2 z_2} = \frac{(4+3\text{i})(-2+2\text{i})}{(-2-2\text{i})(-2+2\text{i})}$$

Using a similar argument to part iii), you can expand the brackets on both the numerator and the denominator and simplify to get:

$$\frac{\bar{z}_1}{\bar{z}_2} = \frac{(4+3\text{i})(-2+2\text{i})}{(-2-2\text{i})(-2+2\text{i})} = \frac{-8+8\text{i}-6\text{i}+6\text{i}^2}{4-4\text{i}+4\text{i}-4\text{i}^2} = \frac{-8+6(-1)+2\text{i}}{4-(-4)} = \frac{-14+2\text{i}}{8}$$

This expression still needs to be written in Cartesian form. You can see that:

$$\frac{\bar{z}_1}{\bar{z}_2} = \frac{-14+2\text{i}}{8} = -\frac{14}{8} + \frac{2\text{i}}{8} = -\frac{7}{4} + \frac{1}{4}\text{i}$$

The complex conjugate of $z_1 + z_3 = 5 - 2\text{i}$ is $\overline{z_1 + z_3} = 5 + 2\text{i}$, which is the same as $\bar{z}_1 + \bar{z}_3$. So $\overline{z_1 + z_3} = \bar{z}_1 + \bar{z}_3$.

The complex conjugate of $z_2 - z_1 = -6 + 5\text{i}$ is $\overline{z_2 - z_1} = -6 - 5\text{i}$, which is the same as $\bar{z}_2 - \bar{z}_1$. So $\overline{z_2 - z_1} = \bar{z}_2 - \bar{z}_1$.

The complex conjugate of $z_3 z_2 = -4$ is $\overline{z_3 z_2} = -4$, which is the same as $\bar{z}_3 \bar{z}_2 = -4$. So $\overline{z_3 z_2} = \bar{z}_3 \bar{z}_2$.

The complex conjugate of $\frac{z_1}{z_2} = -\frac{7}{4} - \frac{1}{4}\text{i}$ is $\overline{\frac{z_1}{z_2}} = -\frac{7}{4} + \frac{1}{4}\text{i}$ which is the same as $\frac{\bar{z}_1}{\bar{z}_2}$. So $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$.

In fact, these are true for any complex numbers *z* and *w*. So you have that $\overline{z+w} = \bar{z} + \bar{w}$, $\overline{z-w} = \bar{z} - \bar{w}$, $\overline{zw} = \bar{z}\bar{w}$ and $\overline{z/w} = \bar{z}/\bar{w}$.

ix) $z_2 + z_3 - z_1 = -5 + 6\text{i}$

You can see from this that $\text{Re}(z_2 + z_3 - z_1) = -5$ and $\text{Im}(z_2 + z_3 - z_1) = 6$.

You can notice that $z_2 + z_3 - z_1 = z_2 - z_1 + z_3$. You can use the answer $z_2 - z_1 = -6 + 5\text{i}$. from part ii) to see that:

$z_2 + z_3 - z_1 = (-6+5\text{i}) + (1+\text{i}) = -6+1+5\text{i}+\text{i} = -5+6\text{i}$.

x) $z_2(z_1 + z_3) = -6 + 14\text{i}$

You can see from this that $\text{Re}(z_2(z_1+z_3)) = -6$ and $\text{Im}(z_2(z_1+z_3)) = 14$.

You can use the answer $z_1 + z_3 = 5 - 2\text{i}$ from part i) to see that

$$z_2(z_1+z_3) = (-2+2\text{i})(5-2\text{i})$$

Expanding the brackets gives:

$$z_2(z_1+z_3) = (-2+2\text{i})(5-2\text{i}) = -10+4\text{i}+10\text{i}-4\text{i}^2$$

You can simplify using the fact that $-4\text{i}^2 = 4$ to get:

$$z_2(z_1+z_3) = -10+4\text{i}+10\text{i}-4\text{i}^2 = -10+4+14\text{i} = -6+14\text{i}.$$

xi) $\dfrac{z_1 z_3}{z_2} = -\dfrac{3}{2} - 2\text{i}$

You can see from this that $\text{Re}(z_1 z_3 / z_2) = -3/2$ and $\text{Im}(z_1 z_3 / z_2) = -2$.

You can use the answer $\dfrac{z_1}{z_2} = -\dfrac{7}{4} - \dfrac{1}{4}\text{i}$ from part iv) and the fact that $\dfrac{z_1 z_3}{z_2} = \dfrac{z_1}{z_2} z_3$ to see that:

$$\frac{z_1 z_3}{z_2} = \left(-\frac{7}{4} - \frac{1}{4}\text{i}\right)(1+\text{i})$$

Expanding the brackets gives:

$$\frac{z_1 z_3}{z_2} = \left(-\frac{7}{4} - \frac{1}{4}\text{i}\right)(1+\text{i}) = -\frac{7}{4} - \frac{7}{4}\text{i} - \frac{1}{4}\text{i} - \frac{1}{4}\text{i}^2$$

You can simplify using the fact that $(-1/4)\text{i}^2 = 1/4$ to get:

$$\frac{z_1 z_3}{z_2} = -\frac{7}{4} - \frac{7}{4}\text{i} - \frac{1}{4}\text{i} - \frac{1}{4}\text{i}^2 = -\frac{7}{4} + \frac{1}{4} - 2\text{i} = -\frac{3}{2} - 2\text{i}.$$

xi) $\dfrac{z_1}{z_2 z_3} = -1 + \dfrac{3}{4}\text{i}$

You can see from this that $\text{Re}(z_1 / z_2 z_3) = -1$ and $\text{Im}(z_1 / (z_2 z_3)) = 3/4$.

You can use the answer $\dfrac{z_1}{z_2} = -\dfrac{7}{4} - \dfrac{1}{4}\text{i}$ from part iv) and the fact that

$\dfrac{z_1}{z_2 z_3} = \dfrac{z_1}{z_2} \div z_3$ to see that:

$$\frac{z_1}{z_2 z_3} = \frac{(-7/4 - 1/4\text{i})}{(1+\text{i})}$$

Remember from the study guide that when you are dividing one complex number by another, you need to make the denominator real. You can do this by multiplying both top and bottom of the division to be done by the complex conjugate of the denominator. You can see that $\bar{z}_3 = 1 - \text{i}$ and so:

$$\frac{z_1}{z_2 z_3} = \frac{z_1 \bar{z}_3}{z_2 z_3 \bar{z}_3} = \frac{(-7/4 - 1/4\text{i})(1-\text{i})}{(1+\text{i})(1-\text{i})}$$

You can expand the brackets on both the numerator and the denominator and simplify to get:

$$\frac{z_1}{z_2 z_3} = \frac{(-7/4 - 1/4\text{i})(1-\text{i})}{(1+\text{i})(1-\text{i})}$$

$$= \frac{-7/4 - 1/4\text{i} + 7/4\text{i} + 1/4\text{i}^2}{1 + \text{i} - \text{i} - \text{i}^2}$$

$$= \frac{-7/4 + 1/4(-1) + 3/2\text{i}}{1-(-1)}$$

$$= \frac{-2 + 3/2\text{i}}{2}$$

This expression still needs to be written in Cartesian form. You can see that

$$\frac{z_1}{z_2 z_3} = \frac{-2 + 3/2\text{i}}{2} = -\frac{2}{2} + \frac{3\text{i}}{2(2)} = -1 + \frac{3}{4}\text{i}$$

xiii) $(\bar{z}_3 \bar{z}_2)(z_3 z_2) = 16$

You can see from this that $\text{Re}(z_1 / z_2 z_3) = 16$ and $\text{Im}(z_1 / z_2 z_3) = 0$.

You can use the answer $z_3 z_2 = -4$ from part iv) and the answer $\bar{z}_3 \bar{z}_2 = -4$ from part vii) to see that

$$(\bar{z}_3\bar{z}_2)(z_3z_2) = (-4)(-4) = 16$$

xiv) $\dfrac{\bar{z}_1}{\bar{z}_2} + \dfrac{\bar{z}_2}{\bar{z}_1} = -\dfrac{231}{100} + \dfrac{17}{100}\text{i}$

You can see from this that $\text{Re}\left(\dfrac{\bar{z}_1}{\bar{z}_2} + \dfrac{\bar{z}_2}{\bar{z}_1}\right) = -\dfrac{231}{100}$ and $\text{Im}\left(\dfrac{\bar{z}_1}{\bar{z}_2} + \dfrac{\bar{z}_2}{\bar{z}_1}\right) = \dfrac{17}{100}$.

You can use the answer $\dfrac{\bar{z}_1}{\bar{z}_2} = -\dfrac{7}{4} + \dfrac{1}{4}\text{i}$ from part iv), but you still need to work out $\dfrac{\bar{z}_2}{\bar{z}_1}$.

You have that:

$$\frac{\bar{z}_2}{\bar{z}_1} = \frac{(-2-2\text{i})}{(4+3\text{i})}$$

Remember from the study guide that when you are dividing one complex number by another, you need to make the denominator real. You can do this by multiplying both top and bottom of the division to be done by the complex conjugate of the denominator. You can see that $z_1 = 4 - 3\text{i}$ and so:

$$\frac{\bar{z}_2}{\bar{z}_1} = \frac{\bar{z}_2 z_1}{\bar{z}_1 z_1} = \frac{(-2-2\text{i})(4-3\text{i})}{(4+3\text{i})(4-3\text{i})}$$

Using a similar argument to part iii), you can expand the brackets on both the numerator and the denominator and simplify to get:

$$\frac{\bar{z}_2}{\bar{z}_1} = \frac{\bar{z}_2 z_1}{\bar{z}_1 z_1} = \frac{(-2-2\text{i})(4-3\text{i})}{(4+3\text{i})(4-3\text{i})} = \frac{-8+6\text{i}-8\text{i}+6\text{i}^2}{16-12\text{i}+12\text{i}-9\text{i}^2} = \frac{-8+6(-1)-2\text{i}}{16-(-9)} = \frac{-14-2\text{i}}{25}$$

This expression still needs to be written in Cartesian form. You can see that

$$\frac{\bar{z}_2}{\bar{z}_1} = \frac{-14-2\text{i}}{25} = -\frac{14}{25} - \frac{2}{25}\text{i}.$$

Finally, you can add together the two parts to get

$$\frac{\bar{z}_1}{\bar{z}_2} + \frac{\bar{z}_2}{\bar{z}_1} = \left(-\frac{7}{4} + \frac{1}{4}\text{i}\right) + \left(-\frac{14}{25} - \frac{2}{25}\text{i}\right) = \left(\frac{-175-56}{100}\right) + \left(\frac{25-8}{100}\right)\text{i} = -\frac{231}{100} + \frac{17}{100}\text{i}$$

xv) $\dfrac{z_1 z_2}{z_2 z_3} = \dfrac{1}{2} - \dfrac{1}{2}\mathrm{i}$

You can see from this that $\mathrm{Re}\left(\dfrac{z_1 z_2}{z_2 z_3}\right) = \dfrac{1}{2}$ and $\mathrm{Im}\left(\dfrac{z_1 z_2}{z_2 z_3}\right) = -\dfrac{1}{2}$.

You can use your answer $z_1 / z_2 z_3 = -1 + 3/4\,\mathrm{i}$ from part xii) and the fact that $z_1 z_2 / z_2 z_3 = (z_1 / z_2 z_3) z_2$ to see that

$$\frac{z_1 z_2}{z_2 z_3} = \left(-1 + \frac{3}{4}\mathrm{i}\right)(-2 + 2\mathrm{i})$$

Expanding the brackets gives:

$$\frac{z_1 z_2}{z_2 z_3} = \left(-1 + \frac{3}{4}\mathrm{i}\right)(-2 + 2\mathrm{i}) = 2 - 2\mathrm{i} - \frac{6}{4}\mathrm{i} + \frac{6}{4}\mathrm{i}^2$$

You can simplify using the fact that $(6/4)\mathrm{i}^2 = -6/4$ to get:

$$\frac{z_1 z_2}{z_2 z_3} = 2 - 2\mathrm{i} + \frac{6}{4}\mathrm{i} + \frac{6}{4}\mathrm{i}^2 = 2 - \frac{6}{4} - \frac{1}{2}\mathrm{i} = \frac{1}{2} - \frac{1}{2}\mathrm{i}$$

xvi) $\dfrac{(z_2 + z_3 - z_1)}{(z_3 + z_1 - z_2)} = -\dfrac{21}{65} + \dfrac{22}{65}\mathrm{i}$

You can see from this that $\mathrm{Re}\left(\dfrac{(z_2 + z_3 - z_1)}{(z_3 + z_1 - z_2)}\right) = -\dfrac{21}{65}$ and $\mathrm{Im}\left(\dfrac{(z_2 + z_3 - z_1)}{(z_3 + z_1 - z_2)}\right) = \dfrac{22}{65}$.

You know that $z_2 + z_3 - z_1 = -5 + 6\mathrm{i}$ from part ix). You can also use your answer $z_1 + z_3 = 5 - 2\mathrm{i}$ from part i) to see that

$$z_3 + z_1 - z_2 = (5 - 2\mathrm{i}) - (-2 + 2\mathrm{i}) = 7 - 4\mathrm{i}.$$

So:

$$\frac{(z_2 + z_3 - z_1)}{(z_3 + z_1 - z_2)} = \frac{(-5 + 6\mathrm{i})}{(7 - 4\mathrm{i})}$$

Remember from the study guide that when you are dividing one complex number by another, you need to make the denominator real. You can do this by multiplying both top and bottom of the division to be done by the complex conjugate of the denominator. You can see that $\overline{z_3 + z_1 - z_2} = 7 + 4\mathrm{i}$ and so:

$$\frac{(z_2+z_3-z_1)\left(\overline{z_3+z_1-z_2}\right)}{(z_3+z_1-z_2)\left(\overline{z_3+z_1-z_2}\right)}=\frac{(-5+6\text{i})(7+4\text{i})}{(7-4\text{i})(7+4\text{i})}$$

Using a similar argument to part iii), you can expand the brackets on both the numerator and the denominator and simplify to get:

$$\frac{(z_2+z_3-z_1)}{(z_3+z_1-z_2)}=\frac{(-5+6\text{i})(7+4\text{i})}{(7-4\text{i})(7+4\text{i})}=\frac{-35-20\text{i}+42\text{i}+24\text{i}^2}{49+28\text{i}-28\text{i}-16\text{i}^2}=\frac{-35+24(-1)+22\text{i}}{49-(-16)}=\frac{-21+22\text{i}}{65}$$

This expression still needs to be written in Cartesian form. You can see that

$$\frac{(z_2+z_3-z_1)}{(z_3+z_1-z_2)}=\frac{-21+22\text{i}}{65}=-\frac{21}{65}+\frac{22}{65}\text{i}\,.$$