Learning Enhancement Team

University of East Anglia
STUDENT SUPPORT SERVICE

Steps into Discrete Mathematics

Basics of Sets

This guide introduces the basics of sets. It is mainly concerned with explaining some of the more common terminology and notation of sets. It discusses subsets, cardinality, and power sets in more detail.

Introduction

One of the most fundamental topics in mathematics is the study of **sets**. A set is **any collection of objects** called individual **elements** (or **members**). For example a set can be a collection of numbers, all the mammals in the world or the things you find in your fridge. Importantly **all the members of a set are different**, so if you have two bottles of milk in your fridge you only need to include "bottle of milk" as an element of the set.

One of the most crucial debates in mathematics was concerned with the definition of what a set was, what you can put in it and what you could then do with those sets. These arguments were deeply philosophical and came about at the end of the 19^{th} century when mathematicians and philosophers tried to write fundamental rules or **axioms** which describe mathematics. Great thinkers such as Georg Cantor, Bertrand Russell, Ernst Zermelo and Kurt Gödel worked towards solving many fundamental problems in set theory and give mathematics a solid framework to build on. This study guide is not concerned with the formulation of the axioms but a **Learning Enhancement Tutor** would be happy to talk to you about them if you want to learn more.

The language and symbols of sets

As explained above, you can think of a set as a collection of objects. Importantly **the order of the objects in the set does not matter**.

A set is a collection of unique objects in any order.

A set is usually denoted by an upper case, italic letter *A*, *B*, *C* and so on, and can be written as a list of its elements, separated by commas and enclosed in curly brackets, if it is reasonable or even possible to do so.

Example: The set A is made up of the elements lion, cat and rat. Write this set in a proper mathematical way.

The set can be properly written as: $A = \{\text{lion, cat, rat}\}$

You can write "lion", "cat" and "rat" in any order and it would still be the set *A*.

A set can also be defined by a rule or property which relates to a general element of that set. General elements in a set are denoted by lower case italic letters such as *a* (for set *A*), *b* (for set *B*) and so on. This is a more technical way of writing a set and is usually reserved for sets of numbers which follow a specific pattern. The definition is still enclosed in curly brackets.

Example: The set B is made up of positive whole numbers that are multiples of 3. Write this set in terms of the general element b.

The set can be written as: $B = \{b : b \text{ is a multiple of } 3, b > 0\}$

This piece of mathematics says that set *B* is made up of general elements denoted by *b*. Specifically, *b* is a multiple of 3 and also greater than 0. The colon is the mathematical symbol for "such that" and links the general element *b* to the restrictions that define the individual elements.

Finally, it is also common for sets which have an obvious pattern to be written using the ellipsis symbol "…" to mean "and so on". As the multiples of 3 are easy to recognise, you can write *B* as:

$$B = \{3, 6, 9, \ldots\}$$

where it has been assumed that the pattern is easily spotted. If you are doubtful that the pattern is easy to see, you should use the more technical way of defining the set using a general element and any restrictions detailed above.

There is a special set that has no elements, this is called the **empty set** (or **null set**) and is given the symbol $\varnothing$.

There are symbols which denote whether something **is an element of** (or **belongs to**) a set or not. For the set *A*, lion is an element of *A* but tiger is not an element of *A*. You can write this mathematically as:

$\text{lion} \in A$ however $\text{tiger} \notin A$

Similarly for the set of positive multiples of 3:

$9 \in B$ however $10 \notin B$

$\in$ is the symbol for "is an element of"

$\notin$ is the symbol for "is not an element of"

Special sets of numbers

The study guide: *Different Kinds of Numbers* describes the history of how and why numbers are classified by today's mathematicians. These classifications are important examples of sets in their own right:

$\mathbb{N}$ the set of **natural** or **counting** numbers

$\mathbb{Z}$ the set of **integers**

$\mathbb{Q}$ the set of **rational** numbers (see study guide: *Types of Fractions*)

$\mathbb{R}$ the set of **real** numbers

$\mathbb{C}$ the set of **complex** numbers (see study guide: *Basics of Complex Numbers*)

Subsets

You can choose elements out of a set to make a new set. This is because the elements you have chosen are still a collection of objects and therefore a set. This new set is called a **subset** of the original set. For example you can choose the members of the cat family out of set *A* and form a subset *C*:

$$C = \{\text{lion, cat}\}$$

You write this in symbols as:

$C \subseteq A$ or equivalently (but more rarely) $A \supseteq C$

The symbol $\subseteq$ means "is a subset of" or "is contained by".

The symbol $\subseteq$ also assumes that the set you are interested in can be a subset of itself, so here $A \subseteq A$. The symbol $\subset$ is used to mean **"is a proper subset of"**, the parent set is not classed as a proper subset of itself.

The symbol $\subset$ means "is a proper subset of".

You can use a strikethrough to modify the subset symbols to mean "is not a subset of" and "is not a proper subset of".

The symbol $\not\subseteq$ means "is not a subset of" or "is not contained by".
The symbol $\not\subset$ means "is not a proper subset of".

For example $A \not\subset A$ means that A is not a proper subset of itself.

By definition, **the empty set is a subset of any set.** So for any set S, $\varnothing \subseteq S$.

The special sets of numbers detailed above are a good illustration of subsets. You can use the subset symbol to write that:

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

This tells you that each natural number is an integer, each integer is a fraction, each fraction is a real number and each real number is a complex number.

Cardinality of a set

It is often important that you know how many elements a set has in it. This is called the **cardinality** of the set.

The **cardinality** of a set is the number of elements in that set.

The **cardinality** of a set S is written as $|S|$.

Example: What are the cardinalities of the sets A, C and $\varnothing$?

As $A = \{\text{lion, cat, rat}\}$, $|A| = 3$
As $C = \{\text{lion, cat}\}$, $|C| = 2$
As $\varnothing$ is the empty set, $|\varnothing| = 0$

Counting the number of elements in sets like these is straightforward as they have a **finite** number of elements. Assigning cardinality to sets like B which have an **infinite** number of elements proves to be more complicated. The mathematician Cantor stated that some sets with an infinite number of elements have a larger cardinality than other such sets. He started off by introducing a symbol **aleph-nought** $\aleph_0$, to denote the cardinality of the set of natural numbers. So:

$|\mathbb{N}| = \aleph_0$ defined as **countably infinite** by Cantor

He then tried to pair up each element of the other special sets of numbers with elements in $\mathbb{N}$. This can be done for the elements $\mathbb{Z}$ and $\mathbb{Q}$ but not $\mathbb{R}$ (and therefore $\mathbb{C}$). Cantor worked out that $\mathbb{Z}$ and $\mathbb{Q}$ have the same cardinality as $\mathbb{N}$ (and are **countable**) but that $\mathbb{R}$ and $\mathbb{C}$ do not (they are **uncountable**). This redefinition of infinities was a great step forward for mathematics and a **Learning Enhancement Tutor** would be happy to discuss this further with you.

Power sets

The final type of set that this guide discusses is the set whose elements comprise **all** the subsets of a particular set. This set is called the **power set**.

> The **power set** of a set is made up from all the subsets of that set.

The power set, of a set *S* for example, is written $P(S)$. A good way to write out a power set is to start with the subset with no elements, then write out all the subsets with one element, then two elements and so on until you write out *S* itself.

Example: Write the power set of $A = \{\text{lion, cat, rat}\}$?

The subset of *A* with no elements is the empty set $\varnothing$.
The subsets of *A* with one element are $\{\text{lion}\}, \{\text{cat}\}$ and $\{\text{rat}\}$.
The subsets of *A* with two elements are $\{\text{lion, cat}\}, \{\text{lion, rat}\}$ and $\{\text{cat, rat}\}$.
The subset with three elements is *A* itself $\{\text{lion, cat, rat}\}$.

Writing these as set gives the power set and so:

$$P(A) = \{\varnothing, \{\text{lion}\}, \{\text{cat}\}, \{\text{rat}\}, \{\text{lion, cat}\}, \{\text{lion, rat}\}, \{\text{cat, rat}\}, A\}$$

You may notice that 1 element of the power set is a set with no elements, 3 elements of the power set are sets with one element, 3 elements of the power set are sets with two elements and 1 element of the power set is a set with three elements. The numbers 1 3 3 1 are also seen in Pascal's triangle because they are generated by the number of ways of picking elements from a set (see factsheet: *Pascal's Triangle*).

If the cardinality of your original set is given by *n*, you generate the n^{th} row of Pascal's Triangle. A property of Pascal's triangle is that if you add together the numbers in the n^{th} row you get 2^n. In other words, **the number of elements in (or cardinality of) the**

power set of a set S with a finite number of elements $|S|$ is given by:

$$|P(S)| = 2^{|S|}$$

Example: What is the cardinality of the power set of $A = \{\text{lion, cat, rat}\}$?

As $|A| = 3$, you can see that:

$$|P(A)| = 2^{|A|} = 2^3 = 8$$

Which you can check by counting the elements in:

$$P(A) = \{\varnothing, \{\text{lion}\}, \{\text{cat}\}, \{\text{rat}\}, \{\text{lion, cat}\}, \{\text{lion, rat}\}, \{\text{cat, rat}\}, A\}$$

Which was calculated in the previous example.

Want to know more?

If you have any further questions about this topic you can make an appointment to see a **Learning Enhancement Tutor** in the **Student Support Service**, as well as speaking to your lecturer or adviser.

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