Learning Enhancement Team

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STUDENT SUPPORT SERVICE

Steps into Discrete Mathematics

Basics of Sequences

This guide introduces the mathematics of sequences and some of the language that describes them. It looks at arithmetic and geometric sequences in more detail.

Introduction

When you write down a list of numbers you are creating something which is called a **sequence** by mathematicians. Each number in the list is called a **term** of that sequence and the terms are separated by commas. Although any list of numbers can be a sequence, usually there are rules which tell you what numbers you can write down. For instance, if you write the **positive integers** (also called the **natural numbers**), starting at 1 and ending at 4 you get the sequence:

$$1, \quad 2, \quad 3, \quad 4$$

The rules would be that:

(i) the sequence starts at 1,
(ii) you get the next term in the sequence by adding 1 to the previous term and
(iii) you stop when you get to 4.

Sequences which stop are called **finite sequences**.

Alternatively you could write down *all* the natural numbers, starting at 1:

$$1, \quad 2, \quad 3, \quad 4, \quad \ldots$$

Here the rules would be:

(i) the sequence starts at 1 and
(ii) you get the next term in the sequence by adding 1 to the previous term.

You use the **ellipsis symbol** ('…' three dots *only*) to indicate that the sequence continues indefinitely. Sequences which do not have an end are called **infinite sequences**. You can also use an ellipsis to fill in the gap of a long sequence. For example the sequence of positive integers from 1 to 20 can be written as:

$$1, \quad 2, \quad 3, \quad 4, \quad \ldots \quad 19, \quad 20$$

Sequences with a pattern

The majority of mathematical sequences follow a strict pattern or recipe, they have a formula for you to follow to generate each term. This recipe is usually given in terms of a number *n* (sometimes *k*) and you generate the terms of the sequence by substituting the positive integers (starting with 1) into the formula for *n*. A sequence which follows a formula is often denoted by s_n where the *s* is for sequence and the subscript *n* tells you the specific term in the sequence.

Example: Write down the infinite sequence defined by $s_n = 3n - 1$.

To find the sequence, input the positive integers into the formula beginning with 1:

When $n = 1$, $3n - 1 = 3 \cdot 1 - 1 = 2$ so $s_1 = 2$
When $n = 2$, $3n - 1 = 3 \cdot 2 - 1 = 5$ so $s_2 = 5$
When $n = 3$, $3n - 1 = 3 \cdot 3 - 1 = 8$ so $s_3 = 8$
When $n = 4$, $3n - 1 = 3 \cdot 4 - 1 = 11$ so $s_4 = 11$ and so on.

The **general term** of the sequence 2, 5, 8, 11, … is $s_n = 3n - 1$.

Arithmetic sequences

You may have noticed in the sequence $s_n = 3n - 1$ that you can make the next term in the sequence by adding 3 to the previous term. It is important you know which number to start with (in the case of $s_n = 3n - 1$, the first term is $s_1 = 2$), but if you do you can generate all the terms in the sequence. This number is often denoted by *a*. A sequence which can be generated by adding the same number, called the **common difference**, to the previous term is called an **arithmetic sequence**. Let's look at the sequence $s_n = 3n - 1$ again:

Value of *n*	1	2	3	4	…	*n*	…
s_n	s_1	s_2	s_3	s_4	…	s_n	…
Term	$2 + 0 \cdot 3$	$2 + 1 \cdot 3$	$2 + 2 \cdot 3$	$2 + 3 \cdot 3$	…	$2 + (n-1) \cdot 3$	…
	2	$2 + 3$	$2 + 6$	$2 + 9$	…		
Sequence	2	5	8	11	…	$3n - 1$	…

The first term is 2 plus 0 times 3, the second term is 2 plus 1 times 3, the third term is 2 plus 2 times 3 and so on. Notice that the multiple of 3 you add is one less than the term number and so the n^{th} term is 2 plus $n - 1$ multiplied by 3. It is true that, for **all** arithmetic sequences, the general or n^{th} term is the first term, *a*, plus $n - 1$ multiplied by the common difference *d*. As an equation:

$$\boxed{s_n = a + (n-1)d}$$

You can use this formula to generate any term in an arithmetic sequence as long as you know the first term and the common difference. This saves a lot of time and effort as you do not need to generate the whole sequence.

Example: What is the 50th term in the sequence $s_n = 3n - 1$?

In the sequence $s_n = 3n - 1$, the first term is $a = 2$ and the common difference is $d = 3$. You need to find s_{50} so $n = 50$. Using the formula:

$$s_{50} = 2 + (50 - 1) \cdot 3 = 149$$

You can also write an arithmetic sequence as $s_{n+1} = s_n + d$ so the next term in the sequence s_{n+1} is generated by adding the common difference to the previous term s_n. When you write a sequence like this, the formula is called a **recurrence relation**.

Geometric sequences

You have seen that to get the next term in an arithmetic sequence you add a number *d* to the previous term. In another common type of sequence, a **geometric sequence**, you **multiply** the previous term by a number called the **common ratio**, which is usually written as *r*.

For example, starting at the number 3 and with $r = 5$, you get:

Value of *n*	1	2	3	4	…	*n*	…
s_n	s_1	s_2	s_3	s_4	…	s_n	…
Term	3	3×5	$3 \times 5 \times 5$	$3 \times 5 \times 5 \times 5 \ldots$			
	3×5^0	3×5^1	3×5^2	3×5^3	…	$3 \times 5^{n-1} \ldots$	
Sequence	3	15	75	375	…		

Each time you multiply the previous term by five and so, for the second term you have multiplied 3 by 5, for the third term, you have multiplied 3 by 5^2 and for the general term you have multiplied 3 by 5^{n-1}. So, in general, the *n*th term can be written as:

$$s_n = ar^{n-1}$$

You can also write a geometric sequence as the recurrence relation $s_{n+1} = rs_n$ so the next term in the sequence s_{n+1} is generated by multiplying the previous term s_n by the common ratio *r*.

Example: The first term of a geometric sequence is 2 and the 7th term 1458. What is the common ratio *r*?

The first term is $a = 2$, and when $n = 7$ (the 7th term) the term is 1458. So, putting these values into the formula for the general term you get:

$$s_7 = 1458 = 2r^{7-1}$$

Dividing by 2 you find that $r^6 = 729$. Taking the sixth root you get:

$$r = \sqrt[6]{729} = 3 \text{ or } -3$$

This mean that there are two geometric sequences which have a first term of 2 and a 7th term of 1458:

Common ratio of 3 gives: 2, 6, 18, 54, 162, 486, 1458, …

Common ratio of -3 gives: 2, -6, 18, -54, 162, -486, 1458, …

Convergence and divergence

An important property of a sequence is if the numbers in the sequence approach a fixed value or not. If the values in a sequence do approach a certain number then the sequence is said to **converge** or the sequence is **convergent**. If a sequence does not converge, it is said to **diverge** or the sequence is **divergent**.

Example: Does the geometric sequence which has $a = \frac{1}{2}$ and $r = \frac{1}{2}$ converge or diverge?

The sequence is $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, … , $\frac{1}{2^n}$ which gets nearer and nearer to zero as *n* gets larger and larger so you can say that this sequence **converges** as it approaches zero as *n* increases.

Example: Does the arithmetic sequence with general term $s_n = 3n - 1$ converge or diverge?

This is the sequence from earlier in this study guide. As you can see, the numbers will keep on getting bigger and bigger as *n* increases. So this sequence **diverges**, in fact it goes to ∞ as *n* approaches ∞. It is common for sequences to approach $\pm\infty$ and every sequence of this kind is divergent.

Example: Does the geometric sequence which has $a = 2$ and $r = -3$ converge or diverge?

This is the sequence 2, -6, 18, -54, 162, -486, 1458, … from the previous section. The numbers get alternately larger and smaller and so it is **divergent**. However the sequence does not approach either $+\infty$ or $-\infty$. This is known as an **alternating sequence** because the numbers are alternately positive and negative.

Other types of sequences

Not all sequences are arithmetic or geometric. For instance you can make a sequence by starting with 1, then another 1 and then making the subsequent terms in the sequence by adding the two previous terms together:

s_n	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	$\ldots$
Sequence	1,	1,	2,	3,	5,	8,	13,	21,	$\ldots$
pattern	given,	given,	$1+1$,	$1+2$,	$2+3$,	$3+5$	$5+8$,	$8+13$,	$\ldots$
			s_1+s_2,	s_2+s_3,	s_3+s_4,	s_4+s_5,	s_5+s_6,	s_7+s_8,	$\ldots$

This is called the **Fibonacci sequence** and is one of the most famous sequences in mathematics. It is named after the Italian mathematician who used it in a book of 1202 but it was known long before this in Indian mathematics. The numbers are deeply connected to nature. For example Fibonacci used the sequence to understand a population of rabbits. The number spirals found in the middle of sunflowers, and other flowers, as well as around pine cones and pineapples for example, are numbers in the Fibonacci sequence. Although the Fibonacci sequence is neither arithmetic nor geometric, it can be described by the recurrence relation:

$$s_n = s_{n-1} + s_{n-2},$$

There are methods to solve recurrence relations and a **Learning Enhancement Tutor** can go through these methods with you. The solution of a recurrence relation is the general term of the sequence. For the Fibonacci sequence:

$$s_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Many sequences have a known pattern but that pattern cannot be written mathematically. For example the sequence:

$$2, \quad 3, \quad 5, \quad 7, \quad 11, \quad 13, \quad \ldots$$

is the sequence of the **prime numbers** but these numbers do not follow any known mathematical pattern. In fact mathematicians have worked on finding any patterns in the prime numbers for many centuries.

Table of common sequences

Name	Sequence	General term	Recurrence relation	Converges?
Arithmetic	$a, a+d, a+2d, \ldots$	$a+(n-1)d$	$s_n = s_{n-1} + d$	When $d = 0$
Geometric	$a, ar, ar^2, ar^3, \ldots$	ar^{n-1}	$s_n = rs_{n-1}$	When $\lvert r \rvert < 1$
± 1 Sequence	$-1,\ 1,\ -1,\ 1,\ \ldots$	$(-1)^n$	$s_n = -s_{n-1}$	No
Uniform	$a,\ a,\ a,\ a,\ a, \ldots$	a	$s_n = a$	Yes
Fibonacci	1, 1, 2, 3, 5, …	$\dfrac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$	$s_n = s_{n-1} + s_{n-2}$	No
Prime	2, 3, 5, 7, 11, …	None	None	No

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