

Steps into Vectors

The Dot Product

This guide introduces the dot product between two vectors. The dot product is alternatively known as the scalar product. This guide describes how it is calculated and what it is used for.

Introduction

The study guide: [Basic Operations with Vectors](#) described how vectors can be added together, subtracted from each other or multiplied by a scalar. Multiplying two vectors together is not something that can be achieved, however there are operations between two vectors that use the language and symbols of multiplication. Specifically these are finding the **dot product** (often called the **scalar product**) and finding the **cross product** (often called the **vector product**) of the two vectors. You can see that word “product” has been used, you will also see that a dot or a cross is commonly used to denote these operations. This all implies multiplication of the vectors, but dot and cross products are not multiplication. This study guide is concerned with the dot product of two vectors. You can read more about the cross product in the study guide: [The Cross Product](#). You should read the study guide: [Basics of Vectors](#) before continuing on with this guide if you are not familiar with calculating the **magnitude of a vector** or with the **rectangular unit vectors**.

How to calculate the dot product

The dot product can be defined one of two ways and it is common for it to be first introduced as follows.

For any two vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ the dot product of them is:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta \quad (\text{say “a dot b”})$$

Where θ is the angle between the two vectors

$|\mathbf{a}|$ is the magnitude of \mathbf{a} , often written as a : $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

$|\mathbf{b}|$ is the magnitude of \mathbf{b} , often written as b : $|\mathbf{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$

You should notice that the \cdot symbol is used, the same as is often used to denote the

multiplication of numbers. You may also notice that $|\mathbf{a}|$, $|\mathbf{b}|$ and $\cos\theta$ are all scalars (numbers) and so the result of the dot product is also a **scalar**. This is why the dot product is often referred to as the scalar product.

The result of finding the dot product of two vectors is a scalar quantity.

This also means that $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$, you can do the dot product either way around. Mathematically you say that the dot product **commutes** (this is not true of the cross product).

You can calculate the dot product of two vectors this way, only if you know the angle θ between them. This limits the use of this particular formula as often you do not know the angle between two vectors. However you can use this formula in important, special cases: when two vectors are **orthogonal** to each other (in other words when they are **at right-angles** to each other) and also to find the dot product of a vector with itself (in other words when the **angle between them is zero**).

Example: What is the dot product of any two vectors that are orthogonal?

The dot product between two vectors (say \mathbf{a} and \mathbf{b}) is:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

If \mathbf{a} and \mathbf{b} are orthogonal, the angle between them is 90° . As $\cos 90^\circ = 0$:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos 90^\circ = 0$$

The dot product of two orthogonal vectors is zero.

This proves to be very useful as you may often be asked to find a **normal vector** (a normal vector is one a right angles to another vector or surface) or to show that two vectors are normal to each other.

You can also use this to learn about the dot product of the **rectangular unit vectors** \mathbf{i} , \mathbf{j} and \mathbf{k} . As \mathbf{i} , \mathbf{j} and \mathbf{k} are orthogonal to each other, the dot product of any one of them with any other is zero.

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = 0$$

$$\mathbf{i} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$$

$$\mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{j} = 0$$

Example: What is the dot product of a vector with itself?

The angle between a vector and itself is zero degrees and so, since $\cos 0 = 1$:

$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}||\mathbf{a}|\cos 0 = |\mathbf{a}|^2 = a^2 = a_1^2 + a_2^2 + a_3^2$$

The dot product of a vector with itself is the square of the magnitude of the vector.

Again you can apply this idea to find something out about the rectangular unit vectors. As they are unit vectors, their magnitude is 1 and, as the square of 1 is 1:

$$\mathbf{i} \cdot \mathbf{i} = |\mathbf{i}||\mathbf{i}|\cos 0 = 1$$

$$\mathbf{j} \cdot \mathbf{j} = |\mathbf{j}||\mathbf{j}|\cos 0 = 1$$

$$\mathbf{k} \cdot \mathbf{k} = |\mathbf{k}||\mathbf{k}|\cos 0 = 1$$

You now have enough information to find an alternative, and extremely important way of calculating the dot product of two vectors. You can think of $\mathbf{a} \cdot \mathbf{b}$ as a process similar to opening the brackets:

$$\mathbf{a} \cdot \mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$$

Here the coefficients of the rectangular unit vectors are simply multiplied together as they are scalars and you take the relevant dot products of the rectangular unit vectors. In this respect the dot product behaves like multiplication (technically it is called **distributivity**) but the actual calculation of the dot product is not multiplication as mentioned above. After opening the brackets you get:

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= a_1b_1\mathbf{i} \cdot \mathbf{i} + a_1b_2\mathbf{i} \cdot \mathbf{j} + a_1b_3\mathbf{i} \cdot \mathbf{k} \\ &+ a_2b_1\mathbf{j} \cdot \mathbf{i} + a_2b_2\mathbf{j} \cdot \mathbf{j} + a_2b_3\mathbf{j} \cdot \mathbf{k} \\ &+ a_3b_1\mathbf{k} \cdot \mathbf{i} + a_3b_2\mathbf{k} \cdot \mathbf{j} + a_3b_3\mathbf{k} \cdot \mathbf{k}\end{aligned}$$

Using the results of the rectangular unit vectors above: any term with the dot product of two different rectangular unit vectors is zero and also the dot product of a rectangular unit vector with itself is 1 gives:

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

This formula is an important, alternative way of calculating the dot product of two vectors.

Example: What is the dot product of $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$?

Here $a_1 = 3$, $a_2 = 2$, $a_3 = 4$, $b_1 = 1$, $b_2 = -2$ and $b_3 = 3$ so:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = 3 \cdot 1 + 2 \cdot (-2) + 4 \cdot 3 = 11$$

Which tells you that the dot product of $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ is 11.

Example: What does the dot product of the vectors $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ tell you?

Here $a_1 = 3$, $a_2 = 2$, $a_3 = 4$, $b_1 = 4$, $b_2 = -2$ and $b_3 = -2$ so:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = 3 \cdot 4 + 2 \cdot (-2) + 4 \cdot (-2) = 0$$

As the dot product of $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ is 0, the two vectors are orthogonal to each other.

Using the dot product to find the angle between two vectors

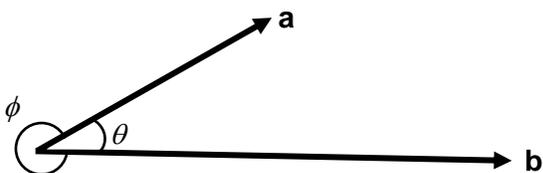
You have seen above that you can calculate the dot product in two ways:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta \quad \text{and also} \quad \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

You can set the right hand sides of these two equations equal to each other as they are both equal to $\mathbf{a} \cdot \mathbf{b}$:

$$|\mathbf{a}||\mathbf{b}|\cos\theta = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Which you can rearrange to find the angle θ between the vectors \mathbf{a} and \mathbf{b} .



The diagram to the left shows that there are in fact **two** angles between \mathbf{a} and \mathbf{b} , θ and ϕ . However $\cos\theta = \cos\phi$ so the dot product cannot distinguish between them. Convention in mathematics is that you choose the smaller

of the angles (θ) to be the angle between \mathbf{a} and \mathbf{b} . For a detailed explanation of why

$\cos \theta = \cos \phi$ see study guide: [Solving Trigonometric Equations](#).

Example: What is the angle between the vectors $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$?

You saw from an earlier example that $a_1b_1 + a_2b_2 + a_3b_3 = 11$. So:

$$|\mathbf{a}||\mathbf{b}|\cos\theta = 11$$

You now only need to calculate the magnitudes of \mathbf{a} and \mathbf{b} , substitute them into the equation and rearrange the result for the angle θ .

$$|\mathbf{a}| = \sqrt{3^2 + 2^2 + 4^2} = \sqrt{29}$$

$$|\mathbf{b}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$$

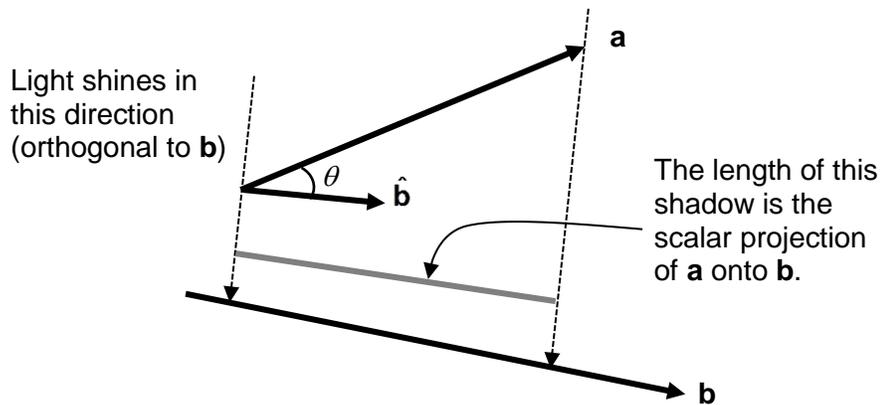
So: $\sqrt{29}\sqrt{14}\cos\theta = 11$ or $\cos\theta = \frac{11}{\sqrt{406}}$

Giving: $\theta = \cos^{-1}\left(\frac{11}{\sqrt{406}}\right) = 56.91^\circ$ or 303.09° to 2 decimal places.

Usually you would choose $\theta = 56.91^\circ$ not the reflex angle 303.09° .

Using the dot product to find the scalar projection of one vector onto another

You can use the dot product to find the **scalar projection** of one vector onto another. You can think of the scalar projection as the length of the shadow that would be cast by one vector onto another by a light shone at right angles to the second vector. This is probably best illustrated by the following diagram of the projection of \mathbf{a} onto \mathbf{b} :



To find the scalar projection you use the dot product of \mathbf{a} and the **unit vector** in the direction of \mathbf{b} , $\hat{\mathbf{b}}$, remembering that the magnitude of a unit vector is 1.

$$\text{Scalar Projection of } \mathbf{a} \text{ onto } \mathbf{b} \text{ is: } \mathbf{a} \cdot \hat{\mathbf{b}} = |\mathbf{a}| \cos \theta$$

Example: What is the scalar projection of $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ onto $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$?

The scalar projection of \mathbf{a} onto \mathbf{b} is given by $\mathbf{a} \cdot \hat{\mathbf{b}} = |\mathbf{a}| \cos \theta$. In the previous example you saw that:

$$\cos \theta = \frac{11}{\sqrt{406}} \quad \text{and} \quad |\mathbf{a}| = \sqrt{29}$$

So:

$$\mathbf{a} \cdot \hat{\mathbf{b}} = |\mathbf{a}| \cos \theta = \frac{11}{\sqrt{406}} \cdot \sqrt{29} = 2.94 \text{ to 2 decimal places.}$$

Often you will need to find the **vector projection** (this is commonly called just the **projection**) of one vector onto another. To do this you multiply the scalar projection by the unit vector you are projecting onto.

$$\text{Vector projection of } \mathbf{a} \text{ onto } \mathbf{b} \text{ is: } (\mathbf{a} \cdot \hat{\mathbf{b}})\hat{\mathbf{b}} = (|\mathbf{a}| \cos \theta)\hat{\mathbf{b}}$$

Want to know more?

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- 💻 Ask: ask.let@uea.ac.uk
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