Learning Enhancement Team

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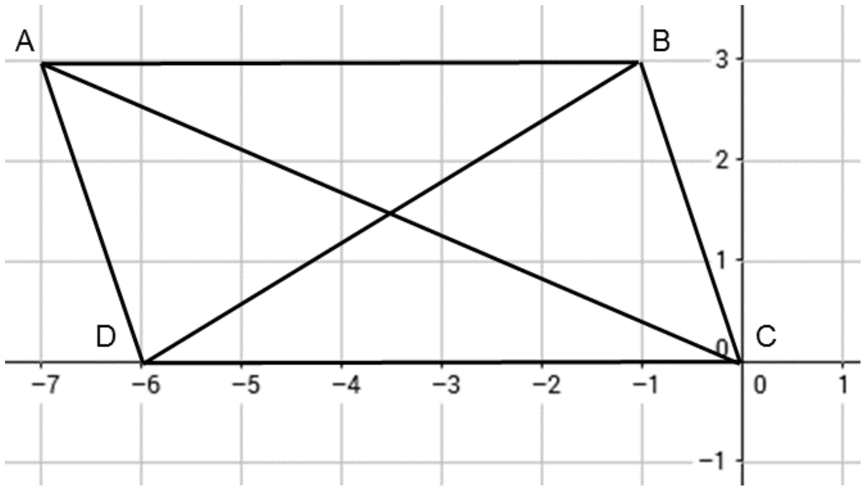
STUDENT SUPPORT SERVICE

Model Answers: The Dot Product

These are the model answers for the worksheet that has questions on the dot product.

The Dot Product study guide

1.



a. The vectors $\overrightarrow{CD}$, $\overrightarrow{CA}$ and $\overrightarrow{CB}$ expressed in terms of rectangular unit vectors **i** and **j** are given by:

$\overrightarrow{CD} = -6\mathbf{i} + 0\mathbf{j}$,

$\overrightarrow{CA} = -7\mathbf{i} + 3\mathbf{j}$ and

$\overrightarrow{CB} = -1\mathbf{i} + 3\mathbf{j}$.

b.

i. The dot product of two vectors **a** and **b** in **2 Dimensional** (2D) space is given by:

$$\mathbf{a}\cdot\mathbf{b} = a_1 b_1 \ + \ a_2 b_2$$

You can think of $\overrightarrow{CD}$ as **a** and $\overrightarrow{CA}$ as **b**.

Then, $a_1 = -6$, $a_2 = 0$, $b_1 = -7$ and $b_2 = 3$.

So, the dot product of $\overrightarrow{CD}$ and $\overrightarrow{CA}$ is:

$$\overrightarrow{CD}\cdot\overrightarrow{CA} = a_1 b_1 \ + \ a_2 b_2 \ = -6\cdot(-7) \ + 0\cdot 3 = 42$$

ii. In order to find the dot product of $\overrightarrow{CD}$ and $\overrightarrow{BA}$ you first need to express $\overrightarrow{BA}$ in terms of the rectangular unit vectors $\mathbf{i}$ and $\mathbf{j}$.

$$\begin{aligned}\overrightarrow{BA} &= \overrightarrow{BC} + \overrightarrow{CA}\\ &= -\overrightarrow{CB} + \overrightarrow{CA}\\ &= -(-1\mathbf{i} + 3\mathbf{j}) + (-7\,\mathbf{i} + 3\mathbf{j})\\ &= (+1-7)\mathbf{i} + (-3+3)\mathbf{j}\\ &= -6\,\mathbf{i} + 0\mathbf{j}\end{aligned}$$

The dot product of two vectors **a** and **b** in **2 Dimensional** (2D) space is given by:

$$\mathbf{a}\cdot\mathbf{b} = a_1 b_1 \ + \ a_2 b_2$$

You can think of $\overrightarrow{CD}$ as **a** and $\overrightarrow{BA}$ as **b**. Then, $a_1 = -6$, $a_2 = 0$, $b_1 = -6$ and $b_2 = 0$.

So, the dot product of $\overrightarrow{CD}$ and $\overrightarrow{BA}$ is:

$$\overrightarrow{CD}\cdot\overrightarrow{BA} = a_1 b_1 \ + \ a_2 b_2 \ = -6\cdot(-6) \ + 0\cdot 0 = 36$$

iii. The dot product two vectors **a** and **b** in **2 Dimensional** (2D) space is given by:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2$$

You can think of $\overrightarrow{CD}$ as **a** and $\overrightarrow{CB}$ as **b**. Then, $a_1 = -6$, $a_2 = 0$, $b_1 = -1$ and $b_2 = 3$.

So, the dot product of $\overrightarrow{CD}$ and $\overrightarrow{CB}$ is:

$$\overrightarrow{CD} \cdot \overrightarrow{CB} = a_1 b_1 + a_2 b_2 = -6 \cdot (-1) + 0 \cdot 3 = 6$$

c. First you need to find the magnitude of the vectors $\overrightarrow{CD}$ and $\overrightarrow{CB}$. The magnitude of a vector in **2 Dimensional space** (2D) is given by: $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$.

So,

$$\left|\overrightarrow{CD}\right| = \sqrt{(-6)^2 + (0)^2} = \sqrt{36 + 0} = \sqrt{36} = 6$$

and $\left|\overrightarrow{CB}\right| = \sqrt{(-1)^2 + (3)^2} = \sqrt{1 + 9} = \sqrt{10}$.

Using the magnitudes and the dot product from (1iib) you need to find the angle (in degrees) between the vectors $\overrightarrow{CD}$ and $\overrightarrow{CB}$.

You know that $\overrightarrow{CD} \cdot \overrightarrow{CB} = \left|\overrightarrow{CD}\right| \left|\overrightarrow{CB}\right| \cos\theta$ where θ is the angle between the two vectors $\overrightarrow{CD}$ and $\overrightarrow{CB}$.

You need to solve $\overrightarrow{CD} \cdot \overrightarrow{CB} = \left|\overrightarrow{CD}\right| \left|\overrightarrow{CB}\right| \cos\theta$ for $\cos\theta$

$\overrightarrow{CD} \cdot \overrightarrow{CB} = \left|\overrightarrow{CD}\right| \left|\overrightarrow{CB}\right| \cos\theta$ is:

$6 = 6 \cdot \sqrt{10} \cos\theta$.

So, $\cos\theta = \sqrt{10}/10$ and $\theta = 71.57°$ in 2 d.p. (To remind yourself of how to solve trigonometric equations you can see the study guide: *Solving Trigonometric Equations*.)

2. The dot product $\mathbf{a}\cdot\mathbf{b}$ is:

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= |\mathbf{a}||\mathbf{b}|\cos\theta \\ &= 5\sqrt{2}\cos(45°) \\ &= 5\cdot\sqrt{2}\cdot\frac{\sqrt{2}}{2} = 5\end{aligned}$$

3. For two vectors to be orthogonal, the angle between them is 90°. As $\cos 90° = 0$ their dot product has to be equal to zero.

a. The dot product two vectors **a** and **b** in **3 Dimensional** (3D) space is given by:

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

The vectors are $\mathbf{a} = -2\mathbf{i} - 5\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} - 1\mathbf{j} + 3\mathbf{k}$.

And so $a_1 = -2$, $a_2 = -5$, $a_3 = 1$, $b_1 = 4$, $b_2 = -1$ and $b_3 = 3$.

So the dot product is:

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = (-2)\cdot 4 + (-5)\cdot(-1) + 1\cdot 3 = -8 + 5 + 3 = 0$$

The two vectors $(-2, -5, 1)$ and $(4, -1, 3)$ are orthogonal, because they have dot product equal to zero.

b. The dot product two vectors **a** and **b** in **3 Dimensional** (3D) space is given by

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

The vectors are $\mathbf{a} = \frac{1}{2}\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = -4\mathbf{i} + 2\mathbf{j} - 1\mathbf{k}$.

And so $a_1 = \frac{1}{2}$, $a_2 = -6$, $a_3 = 2$, $b_1 = -4$, $b_2 = 2$ and $b_3 = -1$.

So the dot product is:

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= a_1b_1 + a_2b_2 + a_3b_3\\ &= \tfrac{1}{2}\cdot(-4)+(-6)\cdot(2)+2\cdot(-1)\\ &= -2-12-2=-16\end{aligned}$$

The two vectors $\left(\tfrac{1}{2},-6,2\right)$ and $(-4,2,-1)$ are not orthogonal, because their dot product is not equal to zero.

c. The dot product two vectors **a** and **b** in **3 Dimensional** (3D) space is given by:

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

The vectors are $\mathbf{a}=1\mathbf{i}-\tfrac{3}{2}\mathbf{j}+\tfrac{1}{8}\mathbf{k}$ and $\mathbf{b}=0\mathbf{i}+\tfrac{1}{4}\mathbf{j}+3\mathbf{k}$.

And so $a_1=1$, $a_2=-\tfrac{3}{2}$, $a_3=\tfrac{1}{8}$, $b_1=0$, $b_2=\tfrac{1}{4}$ and $b_3=3$.

So the dot product is:

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= a_1b_1 + a_2b_2 + a_3b_3\\ &= 1\cdot 0+\left(-\tfrac{3}{2}\right)\cdot\tfrac{1}{4}+\tfrac{1}{8}\cdot(3)\\ &= 0-\tfrac{3}{8}+\tfrac{3}{8}\\ &= 0\end{aligned}$$

The two vectors $\left(1,-\tfrac{3}{2},\tfrac{1}{8}\right)$ and $\left(0,\tfrac{1}{4},3\right)$ are orthogonal, because they have dot product equal to zero.

d. The dot product two vectors **a** and **b** in **3 Dimensional** (3D) space is given by:

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

The vectors are $\mathbf{a}=-2\mathbf{i}+\tfrac{2}{3}\mathbf{j}+\tfrac{\sqrt{5}}{3}\mathbf{k}$ and $\mathbf{b}=1\mathbf{i}-1\mathbf{j}+\sqrt{5}\,\mathbf{k}$.

And so $a_1=-2$, $a_2=\tfrac{2}{3}$, $a_3=\tfrac{\sqrt{5}}{3}$, $b_1=1$, $b_2=-1$ and $b_3=\sqrt{5}$.

So the dot product is:

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$
$$= -2\cdot 1 + \tfrac{2}{3}\cdot(-1) + \tfrac{\sqrt{5}}{3}\cdot\sqrt{5}$$
$$= -2 - \tfrac{2}{3} + \tfrac{5}{3}$$
$$= -1$$

The two vectors $\left(-2, \tfrac{2}{3}, \tfrac{\sqrt{5}}{3}\right)$ and $\left(1, -1, \sqrt{5}\right)$ are not orthogonal, because their dot product is not equal to zero.

4.

a. The vectors $\mathbf{a}$ and $\mathbf{b}$ expressed in terms of rectangular unit vectors are given by:

$$\mathbf{a} = \mathbf{i} + 2\mathbf{j} \text{ and } \mathbf{b} = 3\mathbf{i} + 2\mathbf{j}.$$

b. To find the unit vector of **b** first you need to find the magnitude of **b**. The magnitude of a vector in **2 Dimensional space** (2D) is given by: $|\mathbf{b}| = \sqrt{b_1^2 + b_2^2}$.

So,

$$|\mathbf{b}| = \sqrt{b_1^2 + b_2^2} = \sqrt{3^2 + 2^2} = \sqrt{9+4} = \sqrt{13}\,.$$

Then, the unit vector of **b** is:

$$\hat{\mathbf{b}} = \frac{\widehat{b}_1}{|\mathbf{b}|}\mathbf{i} + \frac{\widehat{b}_2}{|\mathbf{b}|}\mathbf{j} = \frac{3}{\sqrt{13}}\mathbf{i} + \frac{2}{\sqrt{13}}\mathbf{j} = \frac{3\sqrt{13}}{13}\mathbf{i} + \frac{2\sqrt{13}}{13}\mathbf{j}\,.$$

c. The scalar projection of **a** onto **b** is given by $\mathbf{a}\cdot\hat{\mathbf{b}}$. From 4b you have that:

$$\hat{\mathbf{b}} = \frac{3\sqrt{13}}{13}\mathbf{i} + \frac{2\sqrt{13}}{13}\mathbf{j}$$

The dot product of two vectors **a** and $\hat{\mathbf{b}}$ in **2 Dimensional** (2D) space is given by:

$$\mathbf{a}\cdot\widehat{\mathbf{b}} = a_1\hat{b}_1 + a_2\widehat{b}_2$$

Then, $a_1 = 1$, $a_2 = 2$, $b_1 = \frac{3\sqrt{13}}{13}$ and $b_2 = \frac{2\sqrt{13}}{13}$.

So, the dot product is:

$$\mathbf{a} \cdot \hat{\mathbf{b}} = 1 \cdot \frac{3\sqrt{13}}{13} + 2 \cdot \frac{2\sqrt{13}}{13} = \frac{7\sqrt{13}}{13} = 1.94 \text{ in 2 d.p.}$$

The vector projection of **a** onto **b** is $(\mathbf{a} \cdot \hat{\mathbf{b}})\hat{\mathbf{b}}$.

In the previous parts you found that $\mathbf{a} \cdot \hat{\mathbf{b}} = 1.94$ and $\hat{\mathbf{b}} = \frac{3\sqrt{13}}{13}\mathbf{i} + \frac{2\sqrt{13}}{13}\mathbf{j}$.

So, the vector projection is:

$$(\mathbf{a} \cdot \hat{\mathbf{b}})\hat{\mathbf{b}} = \frac{7\sqrt{13}}{13}\left(\frac{3\sqrt{13}}{13}\mathbf{i} + \frac{2\sqrt{13}}{13}\mathbf{j}\right) = \frac{21}{13}\mathbf{i} + \frac{14}{13}\mathbf{j}.$$