Learning Enhancement Team

University of East Anglia
STUDENT SUPPORT SERVICE

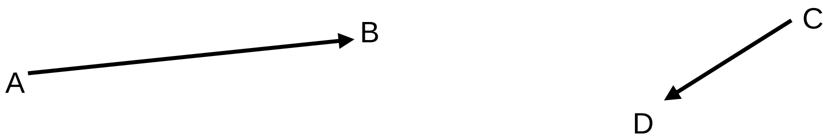
Steps into Vectors

Basics of Vectors

This guide introduces vectors and some of the language that describes them. It defines the magnitude of a vector and how to calculate it. It also introduces unit vectors.

Introduction

Vectors are a very important idea in all areas of science. For example most of modern physics relies on vectors and the theory that surrounds them. **A vector can be thought of as describing a journey from one point to another** and in fact the word vector comes from the Latin for a passenger or carrier. They are usually drawn as a straight arrow joining two points with the arrow showing the direction of travel. For example:

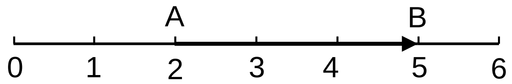


The diagram above shows two vectors one which goes from point A to point B and one which goes from C to D. In mathematics these journeys are written as $\overrightarrow{AB}$ and $\overrightarrow{CD}$. For vectors **the direction is very important** for example the journey from B to A, $\overrightarrow{BA}$ (and also from D to C, $\overrightarrow{DC}$), is a **different** but related vector. You should also see that the distance from A to B *is the same as* the distance from B to A and this is *not* a vector. In mathematics something which has a **magnitude** but not a direction is called a **scalar**. Vectors and scalars are related but different.

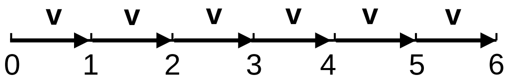
Vectors have magnitude and direction
Scalars have only magnitude

Vectors and scalars

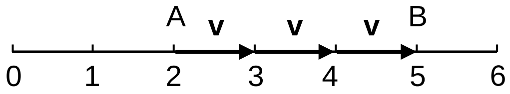
It is crucial that you understand the difference between a vector and a scalar. You can use the number line to help you understand this difference. Take the vector $\overrightarrow{AB}$ above, if A is on the number line at 2 and B is on the number line at 5 then the distance between them is $5-2=3$ and this is a scalar (in fact, all numbers are scalars).



You can also think of the number line as steps of length 1 from left to right. If you think carefully each of these steps is a vector. In mathematics a vector is typed in **bold** (if you are *writing* a vector by hand you should underline it). Let's call the single step along the number line **v** (for vector) so you can re-draw the line as:



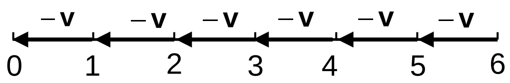
So travelling from A to B (from 2 to 5) uses the vector **v** three times.



You can write that:

$$\overrightarrow{AB} = \mathbf{v} + \mathbf{v} + \mathbf{v} = 3\mathbf{v}$$

You can think of the scalar distance 3 as **scaling** the vector **v** to create the **journey** $\overrightarrow{AB}$. Now think about $\overrightarrow{BA}$ this is the same as $\overrightarrow{AB}$ but in the opposite direction. You need to be careful with this idea, $\overrightarrow{BA}$ and $\overrightarrow{AB}$ are of equal length **but are not the same vector**. One step along the journey $\overrightarrow{AB}$ is **v**. To move one step in the opposite direction, the vector is $-\mathbf{v}$.



For any vector **v**, the reverse journey is defined as $-\mathbf{v}$

So travelling from B to A uses the vector $-\mathbf{v}$ three times and so you can write:

$$\overrightarrow{BA} = (-\mathbf{v}) + (-\mathbf{v}) + (-\mathbf{v}) = 3(-\mathbf{v}) = -3\mathbf{v}$$

Although technically $3(-\mathbf{v})$ is correct; you should write it as $-3\mathbf{v}$ following the SNALPHABET system (see *study guide*: SNALPHABET).

The magnitude of a vector

A special property of a vector is its **magnitude**. Rather than talking about the "length" of a vector, which seems natural, you should actually talk about its magnitude. **The magnitude of a vector is always positive.** In the example above, the magnitude of both $\overrightarrow{AB}$ and $\overrightarrow{BA}$ is 3. In mathematics you will see the magnitude of a vector **v** written in a variety of ways. The most common is $|\mathbf{v}|$ but may also see *v*. In the example above, the magnitude of **v** is 1, which is written as:

$$|\mathbf{v}| = 1 \qquad \text{or} \qquad v = 1$$

This guide, and other guides in this series, uses the $|\mathbf{v}|$ notation instead of any other. The idea of a magnitude is very important and will be more formally defined later in this guide.

Unit vectors

Vectors which have a magnitude equal to 1 have a special name; they are known as **unit vectors**. In mathematics, the number 1 is often referred to as the unit. Any unit vector is given a special symbol, you write the vector with a "hat" on it. For example, as we have seen above **v** has a magnitude of 1 and so is written as $\hat{\mathbf{v}}$ (pronounced "vee-hat"). You can express any vector, for example **a**, as its magnitude $|\mathbf{a}|$ multiplied by its corresponding unit vector $\hat{\mathbf{a}}$. Mathematically this is written as:

$$\mathbf{a} = |\mathbf{a}|\hat{\mathbf{a}}$$

Remember that the magnitude $|\mathbf{a}|$ is a scalar quantity.

The idea of a unit vector is very important as you can use it to describe 3-dimensional space as discussed in the next section of this guide.

Spaces and dimensions

Earlier you saw that the vectors **v** and $-\mathbf{v}$ could be used to move left and right along the number line. If you think of the number line as the ***x*-axis** (horizontal axis) then this is an example of a **1 Dimensional vector space** (1D), in that you can only move left or right along the *x*-axis. By adding the ability to move up and down your space becomes **2 Dimensional** (2D) and you can move along the ***y*-axis**. **It is important that the *x*- and *y*-axes are at right angles to each other**. Finally, you can add a third axis (the ***z*-axis** which is at right angles to the other two) which moves you in and out of the page this makes **3 Dimensional** (3D) space. These 3 axes form a **vector space**. Usually in mathematics you will work in either 2 or 3 dimensional space. At the center of the space is a point where the axes meet, this is known as **the origin**.

Every point in 3D space has a **coordinate** which describes where it is in relation to the origin. In order to get to a particular point in the space from the origin you can take steps along the directions of each axis. The origin has the coordinate $(0,0,0)$ as you take no steps in any direction to get there. If you want to get to the coordinate $(1,2,3)$ you take one step in the positive *x*-direction, two steps in the positive *y*-direction and three steps in the positive *z*-direction. Similarly if you want to get to the point $(-2,0,6)$ you take two steps in the negative *x*-direction, zero steps in the *y*-direction and six steps in the positive *z*-direction and so on. The idea of taking single steps can be exploited, by defining single steps along the respective axes as vectors then you can use them to reach points in space. These vectors are called the **rectangular unit vectors** and are defined as:

i - the unit vector (a single step) in the positive *x*-direction
j - the unit vector (a single step) in the positive *y*-direction
k- the unit vector (a single step) in the positive *z*-direction

The rectangular unit vectors are at right angles to each other (this is called **orthogonal**) and they each have a magnitude of 1.

Example: Use the rectangular unit vectors to describe how to get from the origin to the points: (i) $(1,2,3)$ and (ii) $(-2,0,6)$.

(i) As mentioned above, if you want to get the point $(1,2,3)$ you take:
one step in the positive *x*-direction which is the same as $1\mathbf{i}$, or simply **i**
two steps in the positive *y*-direction which is the same as $2\mathbf{j}$
three steps in the positive *z*-direction which is the same as $3\mathbf{k}$

To write this as a vector you must describe the journey from the origin to $(1,2,3)$. This is one step in the positive *x*-direction written as **i**, followed by two steps in the positive *y*-direction written as $2\mathbf{j}$ and finally three steps in the positive *z*-direction written as $3\mathbf{k}$. You can write this idea as a vector, let's call it **a**, as:

$$\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$$

The vector **a** neatly describes the journey from the origin to the point $(1,2,3)$ as a vector. It is convention to write the steps in the *x*-direction first, followed by those in the *y*-direction and finally those in the *z*-direction. You should get used to seeing vectors written in this format as it is the most common way they are presented.

(ii) As mentioned above, if you want to get the point $(-2,0,6)$ you take:
two steps in the negative *x*-direction which is the same as $-2\mathbf{i}$,
no steps in the *y*-direction which is the same as $0\mathbf{j}$
six steps in the positive *z*-direction which is the same as $6\mathbf{k}$

To write this as a vector you must describe the journey from the origin to $(-2,0,6)$. This is two steps in the negative *x*-direction written as $-2\mathbf{i}$, followed by no steps in the *y*-direction written as $0\mathbf{j}$ and finally six steps in the positive *z*-direction written as $6\mathbf{k}$. You can write this idea as a vector, let's call it **b**, as:

$$\mathbf{b} = -2\mathbf{i} + 0\mathbf{j} + 6\mathbf{k} = -2\mathbf{i} + 6\mathbf{k}$$

If a vector occurs zero times (such as **j** in this example) it is fine to miss it out but you should remember that it occurs zero times.

It is very common for a general vector to be written in terms of **i**, **j** and **k**. For example a general vector **a** is often written:

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

Where a_1 is the number of steps in the *x*-direction, a_2 the number of steps in the *y*-direction and a_3 the number of steps in the *z*-direction.

The magnitude of a vector (revisited)

You can use the values of a_1, a_2 and a_3, in conjunction with Pythagoras' Theorem (see study guide: *Pythagoras' Theorem*) to define the magnitude of a vector as:

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

You should always take the positive square root in this equation. Remember that you are squaring a_1, a_2 and a_3 and that the square of a number is always **positive**. Because of this, a magnitude is always positive and never negative. You can use the equation above to write the corresponding unit vector of **a**, $\hat{\mathbf{a}}$. By rearranging $\mathbf{a} = |\mathbf{a}|\hat{\mathbf{a}}$ you find:

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

By replacing **a** with $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ you get:

$$\hat{\mathbf{a}} = \frac{a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}}{|\mathbf{a}|}$$

It is the convention when writing a vector to break it down into three terms, each comprising a scalar and either **i**, **j** or **k** and so a more correct way of writing $\hat{\mathbf{a}}$ is:

$$\hat{\mathbf{a}} = \frac{a_1}{|\mathbf{a}|}\mathbf{i} + \frac{a_2}{|\mathbf{a}|}\mathbf{j} + \frac{a_3}{|\mathbf{a}|}\mathbf{k}$$

Example: What are the magnitudes and corresponding unit vectors of the vectors (i) $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and (ii) $\mathbf{c} = -2\mathbf{i} + 6\mathbf{k}$?

(i) For the vector $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $b_1 = 1$, $b_2 = 2$ and $b_3 = 3$. Using these values in the equation for the magnitude $|\mathbf{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$ gives:

$$|\mathbf{b}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$$

So the magnitude (or length) of **b** is $\sqrt{14} = 3.74$ to 2 d.p.

The unit vector of **b** is:

$$\hat{\mathbf{b}} = \frac{b_1}{|\mathbf{b}|}\mathbf{i} + \frac{b_2}{|\mathbf{b}|}\mathbf{j} + \frac{b_3}{|\mathbf{b}|}\mathbf{k} = \frac{1}{\sqrt{14}}\mathbf{i} + \frac{2}{\sqrt{14}}\mathbf{j} + \frac{3}{\sqrt{14}}\mathbf{k}$$

You can always check that your answer for a unit vector is correct by showing the magnitude of the vector is 1.

$$|\hat{\mathbf{b}}| = \sqrt{\left(\frac{1}{\sqrt{14}}\right)^2 + \left(\frac{2}{\sqrt{14}}\right)^2 + \left(\frac{3}{\sqrt{14}}\right)^2} = \sqrt{\frac{1}{14} + \frac{4}{14} + \frac{9}{14}} = \sqrt{\frac{1}{1}} = 1$$

(ii) For the vector $\mathbf{c} = -2\mathbf{i} + 6\mathbf{k}$, $c_1 = -2$, $c_2 = 0$ and $c_3 = 6$. Using these values in the equation for the magnitude $|\mathbf{c}| = \sqrt{c_1^2 + c_2^2 + c_3^2}$ gives:

$$|\mathbf{c}| = \sqrt{(-2)^2 + 0^2 + 6^2} = \sqrt{4 + 0 + 36} = \sqrt{40}$$

So the magnitude (or length) of **c** is $\sqrt{40} = 6.32$ to 2 d.p.

The unit vector of **c** is:

$$\hat{\mathbf{c}} = \frac{c_1}{|\mathbf{c}|}\mathbf{i} + \frac{c_2}{|\mathbf{c}|}\mathbf{j} + \frac{c_3}{|\mathbf{c}|}\mathbf{k} = \frac{-2}{\sqrt{40}}\mathbf{i} + \frac{0}{\sqrt{40}}\mathbf{j} + \frac{6}{\sqrt{40}}\mathbf{k} = -\frac{2}{\sqrt{40}}\mathbf{i} + \frac{6}{\sqrt{40}}\mathbf{k}$$

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