Linear First Order Differential Equations

This guide helps you to identify linear first order ordinary differential equations and also to find their solutions.

Introduction

A **differential equation** (or **DE**) is any equation which contains a function and its derivatives, see study guide: *Basics of Differential Equations*. In this guide you can assume that the function under scrutiny is y , and y itself is a function of a variable x. Let's begin with a reminder of the how you can identify a linear first order ordinary differential equation.

Linear:	The function y and any of its derivatives are **only** multiplied by a **constant** or **a function of x**.
First order:	As the order of a DE is determined by the highest order derivative it contains, a **first order DE** is one that contains first derivatives only.
Ordinary:	A DE which contains ordinary derivatives is called an **ordinary differential equation** (or **ODE**).

This guide will help you develop strategies to identify and solve **linear first order ordinary differential equations** *only*. These ODEs can written in general as:

$$y' + yP(x) = Q(x)$$

or alternatively

$$\frac{dy}{dx} + yP(x) = Q(x)$$

The form of the linear first order ODE is strictly: the derivative of y with respect to x plus y multiplied by a function $P(x)$ is equal to $Q(x)$, a different function of x. To use the method of solving linear first order ODEs described in this guide you first need to ensure that your ODE is written exactly as the form above.

Example: Classify the following differential equations and write those that are linear first order ODEs in the form given on the first page of this guide.

(a) $y' + 2xy = e^{-x^2}$ This is a linear first order ODE in the form required where $P(x) = 2x$ and $Q(x) = e^{-x^2}$.

(b) $y'' + 2y = 0$ This is a linear ODE as y and its derivatives are only multiplied by constants. However, it is 2nd order due to the y'' term.

(c) $(y')^2 + e^x y = \sin(x)$ This is a first order ODE but it is nonlinear as y' is squared.

(d) $\frac{dy}{dx} + x\cos(y) = 5x^2$ This is a first order ODE but it is nonlinear since $\cos(y)$ is a nonlinear function of y.

(e) $y' - \cos(x) = y$ This is a linear first order ODE as y and its derivatives are only multiplied by constants. To make this ODE into the form required you need to rearrange like so:

$$y' - y = \cos(x)$$

which shows that $P(x) = -1$ and $Q(x) = \cos(x)$.

Solving first order linear ODEs

1. When $P(x) = 0$

If $P(x) = 0$ in the equations on page 1 of this guide, there is no y term and so you get:

$$\boxed{y' = Q(x)} \quad \text{or alternatively} \quad \boxed{\frac{dy}{dx} = Q(x)}$$

This is a **simple separable differential equation** and can be solved directly by **indefinite integration** (see study guide: *Separable Differential Equations*).

2. When $Q(x) = 0$.

If $Q(x) = 0$ in the equations on page 1 of this guide you get:

$$\boxed{y' + yP(x) = 0} \quad \text{or alternatively} \quad \boxed{\frac{dy}{dx} + yP(x) = 0}$$

This is a more complicated **separable differential equation** which can be solved by

separation of variables (see study guide: *Separable Differential Equations*).
Separating the variables gives:

$$\frac{1}{y}dy = -P(x)dx$$

and after integrating each side you find that $\ln(y) = \int -P(x)dx + c$ which gives:

$$y = Ae^{\int -P(x)dx} \quad (\text{where } A = e^c) \qquad \text{as the } \textbf{general solution} \text{ of } y' + yP(x) = 0$$

You can use this result to find general solutions to ODEs of this type.

Example: Solve $y' + y\sin(x) = 0$.

You can see that this is a linear first order ODE with $P(x) = \sin(x)$ and $Q(x) = 0$. So, after separating the variables and integrating both sides you get the **general solution**:

$$\frac{1}{y}dy = -\sin(x)dx \quad \text{becomes} \quad \ln(y) = \int -\sin(x)dx \quad \text{and so} \quad \ln(y) = \cos(x) + c$$

You can also write the general solution as $y = Ae^{\cos(x)}$ where $A = e^c$. **You can check your answer by differentiation and substitution.** Using the chain rule you find that $y' = -Ae^{\cos(x)}\sin(x)$ and so, using this and the general solution you find that:

$$y' + y\sin(x) = 0$$

Which shows that $y = Ae^{\cos(x)}$ satisfies the ODE $y' + y\sin(x) = 0$.

3. General method for solving $y' + yP(x) = Q(x)$.

The general method for solving linear 1st order ODEs for any $P(x)$ or $Q(x)$ is an example of making a piece of mathematics seemingly more complicated but in fact enables you to simplify it. This may sound confusing but it is common for more complicated mathematics to employ such methods. This particular method uses the chain rule and the product rule from differentiation in an innovative way. Let's look at the left hand side of the general linear first order ODE:

$$y' + yP(x)$$

If you multiply it by $e^{\int P(x)dx}$ (in mathematics this is called the **integrating factor** or **IF**) you get:

$$e^{\int P(x)dx}y' + e^{\int P(x)dx}yP(x)$$

When you first look at this expression you may think that the mathematics has become more complicated. You are right, but now think about the derivative of the integrating factor multiplied by *y*:

$$\frac{d}{dx}\left(ye^{\int P(x)dx}\right) = e^{\int P(x)dx}y' + e^{\int P(x)dx}yP(x)$$

This derivative is found by the product rule, with the chain rule used to integrate the exponential function. Now notice that **the derivative of the IF multiplied by *y* is equal to the left-hand side of the general linear first order ODE multiplied by the IF.** You can use this piece of mathematics to help you solve linear first order ODEs.

Example: Solve $\frac{dy}{dx} + 2xy = e^{-x^2}$.

As discussed on page 2 of this guide, this is a linear first order ODE in the required form where $P(x) = 2x$ and $Q(x) = e^{-x^2}$. To solve the problem you must begin by finding the integrating factor for this ODE:

$$\text{IF} = e^{\int P(x)dx} = e^{\int 2x dx} = e^{x^2}$$

Then multiply both sides of the ODE by this IF to give:

$$e^{x^2}\frac{dy}{dx} + 2xye^{x^2} = 1$$

where, on the right-hand side, $e^{-x^2+x^2} = 1$. You can now use the mathematics explained at the top of this page to see that the left-hand side of the equation can be rewritten as:

$$e^{x^2}\frac{dy}{dx} + 2xye^{x^2} = \frac{d}{dx}\left(e^{x^2}y\right)$$

Using this result on the left-hand side of the problem you get:

$$\frac{d}{dx}\left(e^{x^2}y\right) = 1$$

This form of the ODE can now be integrated with respect to *x*. It is said to be **integrable**.

$$\int \frac{d}{dx}\left(e^{x^2}y\right)dx = \int 1 dx$$

The right-hand side is a basic integral but the left-hand side may be unfamiliar. In fact, the left-hand side is **the integral of a derivative** and, as the two operations are inverses of each other they cancel each other out and you get:

$$e^{x^2}y = x + c$$

Finally, dividing by e^{x^2} you get the **general solution**:

$$y = (x + c)e^{-x^2} \qquad \textbf{general solution of} \qquad \frac{dy}{dx} + 2xy = e^{-x^2}$$

You can check your answer by differentiation and substitution. Using *y* and its derivative $y' = (-2x^2 - 2xc + 1)e^{-x^2}$ in the ODE you can see that:

$$\frac{dy}{dx} + 2xy = e^{-x^2}$$

Showing that the general solution satisfies the differential equation.

Example: Solve $y' - \cos(x) = y$ with the boundary condition $y(0) = 0$.

As discussed on page 2 of this guide, this is a linear first order ODE in the required form where $P(x) = -1$ and $Q(x) = \cos(x)$. Here the integrating factor is:

$$\text{IF} = e^{\int P(x)dx} = e^{\int -1dx} = e^{-x}$$

Then multiply both sides of the ODE by the IF:

$$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}\cos(x)$$

The left-hand side can be expressed as:

$$e^{-x}\frac{dy}{dx} - e^{-x}y = \frac{d}{dx}\left(e^{-x}y\right) \qquad \text{and so the ODE becomes} \qquad \frac{d}{dx}\left(e^{-x}y\right) = e^{-x}\cos(x)$$

Now, integrating both sides (using integration by parts on the right hand side, see study guide: *Integration by Parts*) gives:

$$e^{-x}y = \frac{e^{-x}(-\cos(x) + \sin(x))}{2} + c$$

Finally, dividing by e^{-x} you get the general solution:

$$y = \frac{\sin(x) - \cos(x)}{2} + ce^{x} \qquad \textbf{general solution of} \quad y' - \cos(x) = y$$

You still have an unknown constant *c* in your answer and this can be calculated using the **boundary condition** $y(0) = 0$ given in the question. Remember that *y* is a function of *x*, which can be written as $y(x)$. So the boundary condition $y(0) = 0$ tells you that when $x = 0$, $y = 0$. When you substitute the boundary condition into the general solution you find that:

$$y = \frac{\sin(x) - \cos(x)}{2} + ce^{x} \qquad \text{becomes} \qquad 0 = -\frac{1}{2} + c$$

Or, in other words, $c = 1/2$. You can now substitute this value back into your general solution to give a solution which is **particular** to this boundary condition:

$$y = \frac{\sin(x) - \cos(x) + e^{x}}{2} \qquad \textbf{particular solution of} \quad y' - \cos(x) = y$$

A solution where you have no unknown constants is called a **particular solution**.

Want to know more?

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