Learning Enhancement Team

University of East Anglia
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Model Answers: Linear First Order Differential Equations

These are the model answers for the worksheet that has questions on linear first order differential equations.

Linear First Order Differential Equations worksheet	Linear First Order Differential Equations study guide

1.

a. $$\frac{dy}{dx} = xy + 3$$

This is a **linear** differential equation since the function *y* and all of its derivatives are only multiplied by a constant or a function of *x*.

It is **first order** since the highest order derivative is $\frac{dy}{dx}$, which is first order.

It is **ordinary** since it only contains ordinary derivatives.

You can subtract *xy* from both sides to get:

$$\frac{dy}{dx} - xy = 3$$

The differential equation is now in the form $y' + yP(x) = Q(x)$ which shows that $P(x) = -x$, and $Q(x) = 3$, a constant.

b. $$\frac{dy}{dx} = 5y^2$$

This is a **first order** differential equation since the highest order derivative is $\frac{dy}{dx}$ which is first order.

It is **nonlinear** due to the y^2 term which is a nonlinear function of y.

It is **ordinary** since it only contains ordinary derivatives.

Because it is nonlinear, you cannot write it in the form $y' + yP(x) = Q(x)$

c. $$y\frac{dy}{dx} = 4 + 2x$$

This is a **first order** differential equation since the highest order derivative is $\frac{dy}{dx}$, which is first order.

It is **nonlinear** due to the product of *y* and $\frac{dy}{dx}$.

It is **ordinary** since it only contains ordinary derivatives.

Because it is nonlinear, you cannot write it in the form $y' + yP(x) = Q(x)$

d. $$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

This is a **linear** differential equation since the function *y* and its derivatives are only multiplied by a constant or a function of *x.*

It is **second order** since the highest order derivative is $\frac{d^2y}{dx^2}$, which is a second order derivative.

It is **ordinary** since it only contains ordinary derivatives.

Because it is second order, you cannot write it in the form $y' + yP(x) = Q(x)$

e. $$y' = \frac{\sin(y)}{e^{2x}} + 1$$

This is a **first order** differential equation since the highest order derivative is $\frac{dy}{dx}$, which is first order.

It is **nonlinear** due to the term $\sin(y)$ which is a nonlinear function of *y*.

This is an **ordinary** differential equation since it only contains ordinary derivatives.

Because it is nonlinear, you cannot write it in the form $y' + yP(x) = Q(x)$

f. $$y' + y\cos(x) + 2x = 0$$

This is a **linear** differential equation since the function *y* and its derivatives are only multiplied by a constant or a function of *x*.

Its highest order derivative is y', which is a first order derivative, so it is a **first order** differential equation.

It is **ordinary** since it contains only ordinary derivatives.

You can write it in the form $y' + yP(x) = Q(x)$ by subtracting $2x$ from both sides:

$$y' + y\cos(x) = -2x$$

which shows that $P(x) = \cos(x)$, and $Q(x) = -2x$.

2.

a. $$\frac{dy}{dx} = 2x$$

This differential equation is in the form:

$$\frac{dy}{dx} = Q(x)$$

where $Q(x) = 2x$.

The equation can be read as "the derivative of y with respect to x is equal to $Q(x)$" which looks like the answer to a question in differentiation.

Solving this type of differential equation and finding y involves undoing the differentiation. As indefinite integration is the inverse of differentiation, you can integrate $Q(x)$ with respect to x to find the function y (see study guides: *What is Integration?* and *Integrating Basic Functions*). So:

$$\text{if} \qquad \frac{dy}{dx} = Q(x) \qquad \text{then} \qquad y = \int Q(x)dx$$

In this question, you can find y by integrating $2x$:

$$y = \int 2x\,dx = \frac{2x^2}{2} + c = x^2 + c$$

where c is a constant of integration.

So $\quad y = x^2 + c \quad$ is the **general solution** of $\frac{dy}{dx} = 2x$.

You can check this by differentiating it to get the original DE back.

b. $$\frac{dy}{dx} = \cos(x)$$

This differential equation is in the form:

$$\frac{dy}{dx} = Q(x)$$

where $Q(x) = \cos(x)$, so you can find y by integrating $\cos(x)$:

$$y = \int \cos(x)dx = \sin(x) + c$$

where c is a constant of integration.

So $\quad y = \sin(x) + c \quad$ is the **general solution** of $\quad \frac{dy}{dx} = \cos(x)$.

You can check this by differentiating it to get the original DE back.

c. $$3\frac{dy}{dx} - e^x = 0$$

You can rearrange this ordinary differential equation (ODE) into the form:

$$\frac{dy}{dx} = Q(x)$$

by adding e^x to both sides and then dividing each side by 3 to get:

$$\frac{dy}{dx} = \frac{1}{3}e^x$$

You can then find y by integrating $\frac{1}{3}e^x$:

$$y = \frac{1}{3}\int e^x\, dx = \frac{1}{3}e^x + c$$

where c is a constant of integration.

So $\quad y = \frac{1}{3}e^x + c \quad$ is the **general solution** of $\quad 3\frac{dy}{dx} - e^x = 0$.

You can check this by differentiating it to get the original DE back.

3.

a. $$x\frac{dy}{dx} = 2y$$

This first order ordinary differential equation (ODE) is **separable** (see study guide: *Separable Differential Equations*).

You can divide each side of the equation by x and then divide each side by y to give:

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x}$$

You can then multiply each side of the equation by dx to get:

$$\frac{1}{y}dy = \frac{2}{x}dx$$

The variables are now separated, with functions of y on the left-hand side and functions of x , including any constants, on the right-hand side.

The next step is to integrate both sides of the equation. So:

$$\int \frac{1}{y}dy = \int \frac{2}{x}dx$$

After integrating both sides of the equals sign you get:

$$\ln(y) = 2\ln(x) + c$$

where the constant c results from collecting together the two integration constants introduced from indefinite integration of each side of the equation.

Using the logarithmic transformation (see study guide: *Basics of Logarithms*) you get:

$$e^{\ln(y)} = e^{2\ln(x)+c} \qquad \text{which becomes} \qquad e^{\ln(y)} = e^{\ln(x^2)}e^c$$

Since $2\ln(x) = \ln(x^2)$, and $e^{a+b} = e^a e^b$ (for some arbitrary a and b terms). Then:

$$e^{\ln(y)} = e^c e^{\ln(x^2)} \qquad \text{becomes} \qquad y = Ax^2 \qquad (\text{where } A = e^c).$$

So $y = Ax^2$ is the **general solution** of $x\frac{dy}{dx} = 2y$.

You can check this by differentiating it to get the original DE back.

b. $\frac{dy}{dx} = y\cos(x)$

This first order ordinary differential equation (ODE) is **separable** (see study guide: *Separable Differential Equations*).

You divide each side of the equation by y and then multiply each side of the equation by dx to get:

$$\frac{1}{y}dy = \cos(x)dx$$

The variables have now been separated, with functions of y on the left-hand side and functions of x, including any constants, on the right-hand side.

To find the general solution you integrate both sides of the equation using indefinite integration. So:

$$\int \frac{1}{y} dy = \int \cos(x) dx$$

and after integrating both sides of the equation you get:

$$\ln(y) = \sin(x) + c$$

where the constant *c* results from collecting together the two integration constants from integrating each side of the equation.

Using the logarithmic transformation (see study guide: *Basics of Logarithms*) you get:

$e^{\ln(y)} = e^{\sin(x)+c}$ which becomes $e^{\ln(y)} = e^{\sin(x)} e^{c}$

Then:

$e^{\ln(y)} = e^{c} e^{\sin(x)}$ becomes $y = Ae^{\sin(x)}$ (where $A = e^{c}$)

So $y = Ae^{\sin(x)}$ is the **general solution** to $\frac{dy}{dx} = y\cos(x)$.

You can check this by differentiating it to get the original DE back.

c. $$\frac{dy}{dx} - \frac{4e^{x}}{y} = 0$$

This first order ordinary differential equation (ODE) is **separable** (see study guide: *Separable Differential Equations*).

You can add $\frac{4e^{x}}{y}$ to both sides to give:

$$\frac{dy}{dx} = \frac{4e^{x}}{y}$$

Then you multiply both sides by *y* and then multiply both sides by *dx* to get:

$$y\, dy = 4e^{x} dx$$

The variables are now separated with functions of *y* on the left-hand side and functions of *x*, including any constants, on the right-hand side.

To find the **general solution** you now integrate both sides of the equation. So:

$$\int y\, dy = \int 4e^{x} dx$$

After integrating each side you get:

$$\frac{y^2}{2} = 4e^x + c$$

where the constant *c* results from collecting together the two integration constants from integrating each side of the equation. You can then multiply both sides of the equation by 2 and then take the square root of both sides to give:

$$y = \sqrt{8e^x + 2c}$$

Finally, if you let $c = 2k$, (then $2c = 4k$) where *k* is another arbitrary constant, then you can factorise the terms inside the square root:

$$y = \sqrt{8e^x + 4k} \qquad \text{becomes} \qquad y = \sqrt{4(2e^x + k)}$$

and this can be simplified further:

$$y = \sqrt{4}\sqrt{(2e^x + k)} \qquad \text{which becomes} \qquad y = 2\sqrt{2e^x + k}$$

So $\quad y = 2\sqrt{2e^x + k}$ is the **general solution** of $\frac{dy}{dx} - \frac{4e^x}{y} = 0$.

You can check this by differentiating it to get the original DE back.

4.

a. $$\frac{dy}{dx} + 4xy = e^{-2x^2}$$

This ordinary differential equation (ODE) is in the form $y' + yP(x) = Q(x)$, where

$$P(x) = 4x \qquad \text{and} \qquad Q(x) = e^{-2x^2}.$$

To find the **general solution** to this ODE, you can start by finding an **integrating factor** (**IF**), see study guide: *Linear First Order Differential Equations*.

The integrating factor is defined to be:

$$\mathbf{IF} = e^{\int P(x)dx}$$

so in this question you have:

$$\mathbf{IF} = e^{\int P(x)dx} = e^{\int 4x\,dx} = e^{2x^2}$$

The next step is to multiply both sides of the ODE by this integrating factor to give:

$$e^{2x^2}\frac{dy}{dx} + 4xye^{2x^2} = 1$$

where on the right-hand side, $e^{-2x^2}e^{2x^2} = e^{2x^2 - 2x^2} = e^0 = 1$. You can now use the relation:

$$\frac{d}{dx}\left(e^{2x^2}y\right)=e^{2x^2}\frac{dy}{dx}+4xye^{2x^2}$$

to rewrite $e^{2x^2}\frac{dy}{dx}+4xye^{2x^2}=1$ as:

$$\frac{d}{dx}\left(e^{2x^2}y\right)=1$$

This form of the ODE can now be integrated with respect to x :

$$\int\frac{d}{dx}\left(e^{2x^2}y\right)dx=\int 1dx$$

The right-hand side is a basic integral: $\int 1dx=x+c$.

The left-hand side is the **integral of a derivative,** and as integration and differentiation are inverses of each other they cancel each other out. So you get:

$$e^{2x^2}y=x+c$$

Finally, dividing each side of the equation by e^{2x^2} you get:

$y=e^{-2x^2}(x+c)$ which is the **general solution** of $\frac{dy}{dx}+4xy=e^{-2x^2}$.

b. $$x\frac{dy}{dx}=2y+x^2$$

You can rewrite this ordinary differential equation (ODE) in the form $y'+yP(x)=Q(x)$ First, divide each side by x to give:

$$\frac{dy}{dx}=\frac{2y}{x}+x$$

Then subtract $\frac{2y}{x}$ from each side to get:

$$\frac{dy}{dx}-\frac{2y}{x}=x$$

This is now in the form $y'+yP(x)=Q(x)$ where $P(x)=-\frac{2}{x}$ and $Q(x)=x$.

To find the **general solution** to this ODE, you can start by finding an **integrating factor** (**IF**), see study guide: *Linear First Order Differential Equations*.

The integrating factor is defined to be:

$$\mathbf{IF} = e^{\int P(x)dx}$$

so in this question you have:

$$\mathbf{IF} = e^{\int P(x)dx} = e^{-\int \frac{2}{x}dx} = e^{-2\ln(x)}$$

Then, since $-2\ln(x) = \ln(x^{-2}) = \ln\left(\frac{1}{x^2}\right)$, you can rewrite the integrating factor:

$$\mathbf{IF} = e^{-2\ln(x)} = e^{\ln\left(\frac{1}{x^2}\right)} = \frac{1}{x^2}$$

The next step is to multiply both sides of the ordinary differential by this integrating factor to give:

$$\frac{1}{x^2}\frac{dy}{dx} - \frac{2y}{x^3} = \frac{1}{x}$$

Now you can use the relation:

$$\frac{d}{dx}\left(\frac{y}{x^2}\right) = \frac{1}{x^2}\frac{dy}{dx} - \frac{2y}{x^3}$$

To rewrite $\frac{1}{x^2}\frac{dy}{dx} - \frac{2y}{x^3} = \frac{1}{x}$ as:

$$\frac{d}{dx}\left(\frac{y}{x^2}\right) = \frac{1}{x}$$

This form of the differential equation can now be integrated with respect to x :

$$\int \frac{d}{dx}\left(\frac{y}{x^2}\right)dx = \int \frac{1}{x}dx$$

which, after integrating becomes:

$$\frac{y}{x^2} = \ln(x) + c$$

Finally, multiply each side of the equation by x^2 to get:

$y = x^2(\ln(x) + c)$ which is the **general solution** of $x\frac{dy}{dx} = 2y + x^2$.

c. $y' - \sin(2x) = -\frac{y}{x}$

You can rewrite this ordinary differential equation (ODE) in the form $y' + yP(x) = Q(x)$ by first adding $\sin(2x)$ to both sides of the equation and then adding $\frac{y}{x}$ to each side:

$$y' + \frac{y}{x} = \sin(2x)$$

This is now in the form $y' + yP(x) = Q(x)$ where

$$P(x) = \frac{1}{x} \qquad \text{and} \qquad Q(x) = \sin(2x).$$

To find the **general solution** to this ODE, you can start by finding an **integrating factor** (**IF**), see study guide: *Linear First Order Differential Equations*.

The integrating factor is defined to be:

$$\mathbf{IF} = e^{\int P(x)\,dx}$$

so in this question you have:

$$\mathbf{IF} = e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln(x)} = x$$

The next step is to multiply both sides of the ODE by this integrating factor to get:

$$xy' + y = x\sin(2x)$$

Now you can use the relation:

$$\frac{d}{dx}\left(ye^{\int P(x)dx}\right) = e^{\int P(x)dx}y' + e^{\int P(x)dx}yP(x)$$

to rewrite the left-hand side of the equation as:

$$xy' + y = \frac{d}{dx}(xy)$$

Using this result you can rewrite $xy' + y = x\sin(2x)$ as:

$$\frac{d}{dx}(xy) = x\sin(2x)$$

This form of the ordinary differential equation can now be integrated with respect to *x*. Integrating the left-hand side gives:

$$\int \frac{d}{dx}(xy)dx = xy$$

The right-hand side is slightly more complicated to integrate. You need to use

integration by parts to integrate $x\sin(2x)$ (see study guide: *Integration by Parts*).

The integration by parts formula is given by:

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

Usually, if a power function is part of the integrand it is good practice to let it equal *u*, then $\frac{dv}{dx}$ is the other function by default.

In this example, which contains a power function, it is sensible to make $u = x$ which means that $\frac{dv}{dx} = \sin(2x)$.

To solve this integral you need to find *v* and $\frac{du}{dx}$, and substitute them (along with *u*) into the right-hand side of the integration by parts formula.

You have to differentiate *u* to find $\frac{du}{dx}$ and integrate $\frac{dv}{dx}$ to find *v*. So in this question:

$$u = x \qquad \text{so} \qquad \frac{du}{dx} = 1,$$

$$\frac{dv}{dx} = \sin(2x) \quad \text{so} \qquad v = \int \sin(2x)dx = -\frac{1}{2}\cos(2x)$$

You can then substitute these into the integration by parts formula to get:

$$\int x\sin(2x)dx = -\frac{1}{2}x\cos(2x) + \int \frac{1}{2}\cos(2x)dx$$

Then you evaluate the integral on the right-hand side:

$$\int \frac{1}{2}\cos(2x)dx = \frac{1}{4}\sin(2x) + c$$

and therefore:

$$\int x\sin(2x)dx = -\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x) + c \ .$$

Now you can substitute this back into the ODE:

$$\int \frac{d}{dx}(xy)dx = \int x\sin(2x)dx \qquad \text{becomes} \qquad xy = -\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x) + c$$

Finally, divide each side of the equation by x to give:

$$y=-\frac{1}{2}\cos(2x)+\frac{1}{x}\left(\frac{1}{4}\sin(2x)+c\right)$$

which is the **general solution** of $y'-\sin(2x)=-\frac{y}{x}$.

5.

a. $\frac{dy}{dx}+2xy=e^{-x^2}$ with the initial condition $y(0)=2$

This ordinary differential equation (ODE) is in the form $y'+yP(x)=Q(x)$, where

$P(x)=2x$ and $Q(x)=e^{-x^2}$.

To find the **general solution** to this ODE, you can start by finding an **integrating factor** (**IF**), see study guide: *Linear First Order Differential Equations*.

The integrating factor is defined to be:

$$\textbf{IF}=e^{\int P(x)\,dx}$$

so in this question you have:

$$\textbf{IF}=e^{\int P(x)dx}=e^{\int 2x\,dx}=e^{x^2}$$

The next step is to multiply both sides of the ODE by the integrating factor, which gives:

$$e^{x^2}\frac{dy}{dx}+2xye^{x^2}=e^{-x^2}e^{x^2}$$

and since $e^{x^2}e^{-x^2}=e^0=1$ the equation becomes:

$$e^{x^2}\frac{dy}{dx}+2xye^{x^2}=1$$

Now you can use the relation:

$$\frac{d}{dx}\left(ye^{\int P(x)dx}\right)=e^{\int P(x)dx}y'+e^{\int P(x)dx}yP(x)$$

to rewrite the left-hand side of the equation as:

$$e^{x^2}\frac{dy}{dx}+2xye^{x^2}=\frac{d}{dx}\left(ye^{x^2}\right)$$

Using this result you can rewrite the ODE as:

$$\frac{d}{dx}\left(ye^{x^2}\right)=1$$

The next step is to integrate both sides with respect to x, so:

$$\int \frac{d}{dx}\left(ye^{x^2}\right)dx = \int 1dx$$

and after integrating you get:

$$ye^{x^2} = x + c$$

You can then multiply each side of the equation by e^{-x^2} to get:

$$ye^{x^2}e^{-x^2} = e^{-x^2}(x+c)$$

and since $e^{x^2}e^{-x^2} = e^{x^2-x^2} = e^0 = 1$, the equation becomes

$$y = e^{-x^2}(x+c)$$

which is the **general solution** of $\frac{dy}{dx} + 2xy = e^{-x^2}$.

In order to find the **particular solution** you need to use the initial condition

$y(0)=2$ which tells you that when $x = 0$, then $y = 2$.

You can substitute this into the **general solution**:

$y = e^{-x^2}(x+c)$ becomes $2 = e^0(0+c)$

and since $e^0 = 1$ then $c = 2$. So:

$$y = e^{-x^2}(x+2)$$

is the **particular solution** to $\frac{dy}{dx} + 2xy = e^{-x^2}$ when $y(0)=2$.

b. $x\frac{dy}{dx} = y + x^3$ with the boundary condition $y(1)=5$

You can rewrite this ordinary differential equation (ODE) in the form $y' + yP(x) = Q(x)$ First divide each side by x to give:

$$\frac{dy}{dx} = \frac{y}{x} + x^2$$

Then subtract $\frac{y}{x}$ from each side of the equation to get:

$$\frac{dy}{dx} - \frac{y}{x} = x^2$$

This ODE is now in the form $y' + yP(x) = Q(x)$ where

$$P(x) = -\frac{1}{x} \qquad \text{and} \qquad Q(x) = x^2 .$$

To find the **general solution** to this ODE, you can start by finding an **integrating factor** (**IF**), see study guide: *Linear First Order Differential Equations*.

The integrating factor is defined to be:

$$\mathbf{IF} = e^{\int P(x)dx}$$

so in this question you have:

$$\mathbf{IF} = e^{\int P(x)dx} = e^{-\int \frac{1}{x}dx} = e^{-\ln(x)} = e^{\ln(x^{-1})} = e^{\ln\left(\frac{1}{x}\right)} = \frac{1}{x}$$

The next step is to multiply both sides of the ODE by this integrating factor to get:

$$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = x$$

Now you can use the relation:

$$\frac{d}{dx}\left(ye^{\int P(x)dx}\right) = e^{\int P(x)dx}y' + e^{\int P(x)dx}yP(x)$$

to rewrite the left-hand side of the equation as:

$$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = \frac{d}{dx}\left(\frac{y}{x}\right)$$

Using this result you can rewrite the ODE as:

$$\frac{d}{dx}\left(\frac{y}{x}\right) = x$$

The next step is to integrate both sides with respect to *x* to get:

$$\int \frac{d}{dx}\left(\frac{y}{x}\right)dx = \int x\,dx$$

which, after integrating, becomes

$$\frac{y}{x} = \frac{x^2}{2} + c$$

Then you can multiply each side of the equation by x to get:

$$y = \frac{x^3}{2} + cx$$ which is the **general solution** of $x\frac{dy}{dx} = y + x^3$

To find the **particular solution** you need to use the boundary condition

$y(1) = 5$ which tells you that when $x = 1$, then $y = 5$.

You can substitute this into the **general solution**:

$y = \frac{x^3}{2} + cx$ becomes $5 = \frac{1}{2} + c$,

and rearranging gives $c = \frac{9}{2}$. So

$$y = \frac{1}{2}\left(x^3 + 9x\right)$$

is the **particular solution** to $x\frac{dy}{dx} = y + x^3$, when $y(1) = 5$.

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