Learning Enhancement Team

Model Answers: Separable Differential Equations

These are the model answers for the worksheet that has questions on separable differential equations.

Separable Differential Equations worksheet	Separable Differential Equations study guide

1.

a. $$\frac{dy}{dx} = x$$

You can multiply each side of this equation by dx to give:

$$dy = x dx$$

The variables are separated with functions of y on the left-hand side of the equals sign and functions of x on the right-hand side. So the equation is in the form $g(y) = f(x)$.

b. $$\frac{dy}{dx} = 5y$$

You can divide each side of this equation by y to get:

$$\frac{1}{y}\frac{dy}{dx} = 5$$

and then you can multiply each side of this equation by dx to give:

$$\frac{1}{y} dy = 5 dx$$

The variables are separated with functions of y on the left-hand side of the equals sign and functions of x, including any constants, on the right-hand side. So the equation is in the form $g(y) = f(x)$.

c. $\frac{dy}{dx} = 4 + 2x$

You can multiply each side of this equation by dx to give:

$$dy = (4 + 2x)dx$$

The variables are separated, with functions of y on the left-hand side of the equals sign and functions of x, including any constants, on the right-hand side. So the equation is in the form $g(y) = f(x)$.

d. $\frac{dy}{dx} + 3y = 2x$

It is not possible to separate the variables here. There is another method for solving this equation (see study guide: *Linear First Order Differential Equations*).

e. $\frac{dy}{dx} = \frac{\sin(y)}{e^x}$

You can divide both sides of the equation by $\sin(y)$ and then multiply both sides of the equation by dx to get:

$$\frac{dy}{\sin(y)} = \frac{dx}{e^x}$$

The variables are separated, with functions of y on the left-hand side of the equals sign and functions of x, including any constants, on the right-hand side. So the equation is in the form $g(y) = f(x)$.

f. $x^2 \frac{dy}{dx} - 5 = 2yx$

It is not possible to separate the variables here. There is another method for solving this equation (see study guide: *Linear First Order Differential Equations*).

g. $y\frac{dy}{dx} = \frac{x^2}{y}$

You can multiply each side of the equation by y and then multiply both sides by dx to give:

$$y^2 dy = x^2 dx$$

The variables are separated, with functions of *y* on the left-hand side of the equals sign and functions of *x*, including any constants, on the right-hand side. So the equation is in the form $g(y) = f(x)$.

h. $$\frac{dy}{dx} + y\cos(x) = 0$$

You can subtract $y\cos(x)$ from each side of the equation to get:

$$\frac{dy}{dx} = -y\cos(x)$$

Then you can divide both sides of the equation by *y* and multiply each side of the equation by dx to give:

$$\frac{dy}{y} = -\cos(x)dx$$

The variables are separated, with functions of *y* on the left-hand side of the equals sign and functions of *x*, including any constants, on the right-hand side. So the equation is in the form $g(y) = f(x)$.

i. $$\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}$$

You cannot separate the variables here. In fact, this is a **homogeneous first order differential equation** and requires a special method to solve it (see study guide: *Homogeneous First Order Differential Equations*).

2.

a. $$\frac{dy}{dx} = 6x$$

You can multiply both sides of the equation by dx to give:

$$dy = 6x dx$$

The variables are now separated, with functions of *y* on the left-hand side of the equals sign and functions of *x*, including any constants, on the right-hand side. So the equation is in the form $g(y) = f(x)$.

This differential equation is in the form:

$$\frac{dy}{dx} = f(x) \qquad \text{or, after separating the variables} \qquad dy = f(x)dx$$

where any constants are part of $f(x)$.

The equation can be read as “the derivative of *y* with respect to *x* is equal to $f(x)$” which looks like the answer to a question in differentiation (see study guide: *What is differentiation?*).

Solving this type of differential equation and finding *y* involves undoing the differentiation. As indefinite integration is the inverse of differentiation, if you integrate $f(x)$ with respect to *x* you will find the function *y*, (see study guide: *Integrating Basic Functions*). So:

$$\text{if} \quad \frac{dy}{dx} = f(x) \quad \text{then} \quad \int dy = \int f(x)dx \quad \text{which results in} \quad y = \int f(x)dx$$

going back this question, you can find y by integrating $6x$:

$$y = \int 6x\,dx = \frac{6x^2}{2} + c = 3x^2 + c$$

where *c* is a constant of integration.

So $\quad y = 3x^2 + c \quad$ is the **general solution** of $\frac{dy}{dx} = 6x$

b. $$y^2 \frac{dy}{dx} = e^x$$

You can multiply each side of the equation by dx to give:

$$y^2 dy = e^x dx$$

The variables are now separated, with functions of *y* on the left-hand side of the equals sign and functions of *x*, including any constants, on the right-hand side. So the equation is in the form $g(y) = f(x)$.

To find the **general solution** you integrate both sides of the equation using indefinite integration (see study guide: *Integrating Basic Functions*). So:

$$\int y^2 dy = \int e^x dx$$

After integrating both sides of the equation you get:

$$\frac{y^3}{3} = e^x + c$$

This is the **general solution** to $y^2 \frac{dy}{dx} = e^x$ where the constant *c* results from collecting together the two integration constants from each side of the equation.

Although this is the general solution to the problem, it is useful to rearrange it to give your solution in the form: "$y = ...$".

To do this, first multiply each side of the equation by 3 to get:

$$y^3 = 3(e^x + c)$$

You can then take the cube root of both sides of the equation to get:

$$y = \sqrt[3]{3(e^x + c)}$$

which is the final form of your **general solution** to $y^2 \frac{dy}{dx} = e^x$.

c. $$\frac{1}{\sin(x)} \frac{dy}{dx} = y$$

First, divide each side of the equation by *y* and then multiply each side of the equation by $\sin(x)$ to get:

$$\frac{1}{y} \frac{dy}{dx} = \sin(x)$$

Then multiply each side of the equation by *dx* to give:

$$\frac{1}{y} dy = \sin(x) dx$$

The variables are now separated, with functions of y on the left-hand side of the equals sign and functions of x, including any constants, on the right-hand side. So the equation is in the form $g(y) = f(x)$.

To find the **general solution**, first integrate both sides of the equation using indefinite integration (see study guide: *Integrating Basic Functions*).
So:

$$\int \frac{dy}{y} = \int \sin(x)dx$$

After integrating both sides of the equation you get:

$$\ln(y) = -\cos(x) + c$$

where the constant c results from collecting together the two integration constants introduced from integrating each side of the equation.

Using the logarithmic transformation (see study guide: *Basics of Logarithms*) you get:

$$e^{\ln(y)} = e^{-\cos(x)+c} \qquad \text{which becomes} \qquad y = e^{-\cos(x)} e^{c}$$

and this can be written as $y = Ae^{-\cos(x)}$ (where $A = e^{c}$)

So $y = Ae^{-\cos(x)}$ is the **general solution** to $\frac{1}{\sin(x)} \frac{dy}{dx} = y$.

d. $$2y\frac{dy}{dx} = 4(x^2 + 1)$$

The first step is to separate the variables. To do this you can divide both sides of the equation by 2 and then multiply both sides by dx to give:

$$y\,dy = 2(x^2 + 1)dx$$

The variables are now separated, with functions of y on the left-hand side of the equals sign and functions of x, including any constants, on the right-hand side. So the equation is in the form $g(y) = f(x)$.

To find the **general solution**, you must now integrate both sides of the equation using indefinite integration (see study guide: *Integrating Basic Functions*). So:

$$\int y\,dy = 2\int (x^2 + 1)dx$$

After integrating both sides of the equation you get:

$$\frac{y^2}{2} = 2\left(\frac{x^3}{3} + x\right) + c$$

where the constant *c* results from collecting together the two integration constants from the integrating both sides of the equation.

Then you can multiply out the brackets and multiply each side of the equation by 2 to get:

$$y^2 = \frac{4}{3}x^3 + 4x + 2c = 2\left(\frac{2}{3}x^3 + 2x + c\right)$$

Although this is the **general solution** to the problem, it is useful to rearrange it to give your solution in the form: "$y = \ldots$". Take the square root of both sides of the equation to get:

$y = \sqrt{2\left(\frac{2}{3}x^3 + 2x + c\right)}$ which is the **general solution** of $2y\frac{dy}{dx} = 4\left(x^2 + 1\right)$.

e. $\frac{dy}{dx} = y^2 \cos(x)$

The first step is to separate the variables. To do this you can divide both sides of the equation by y^2 and then multiply both sides by dx to give:

$$\frac{1}{y^2}dy = \cos(x)dx$$

The variables are now separated, with functions of *y* on the left-hand side of the equals sign and functions of *x*, including any constants, on the right-hand side. So the equation is in the form $g(y) = f(x)$.

To find the **general solution**, you must now integrate both sides of the equation using indefinite integration (see study guide: *Integrating Basic Functions*). So:

$$\int \frac{1}{y^2}dy = \int \cos(x)\,dx$$

After integrating both sides of the equation you get:

$$-\frac{1}{y} = \sin(x) + c$$

where the constant *c* results from the adding together the constants introduced from the integrating of each side of the equation. You can then multiply both sides of the equation by *y* to get:

$$-1 = y(\sin(x) + c)$$

Then divide each side of the equation by $\sin(x) + c$ to give:

$$-\frac{1}{\sin(x)+c} = y$$

So $y = -\frac{1}{\sin(x)+c}$ is the **general solution** to $\frac{dy}{dx} = y^2 \cos(x)$.

f. $$(x^2 + 1)\frac{dy}{dx} = 2xy$$

The first step is to separate the variables. To do this you can divide both sides of the equation by $x^2 + 1$ and then divide both sides by y to give:

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2 + 1}$$

then multiply both sides by dx to give:

$$\frac{1}{y}dy = \frac{2x}{x^2 + 1}dx$$

The variables are now separated, with functions of *y* on the left-hand side of the equals sign and functions of *x*, including any constants, on the right-hand side. So the equation is in the form $g(y) = f(x)$.

To find the **general solution**, you must now integrate both sides of the equation using indefinite integration (see study guide: *Integrating Basic Functions*). So:

$$\int \frac{1}{y}dy = \int \frac{2x}{x^2 + 1}dx$$

(**Top tip:** you can recognise here that $2x$ is the derivative of $x^2 + 1$.

Whenever you integrate a function in the form $\frac{f'(x)}{f(x)}$, then the solution to the integral is just $\ln(f(x))$. Or in other words, if you integrate a function where the numerator is the derivative of the denominator, then the solution to the integral is the natural logarithm of the denominator. See study guide: *Integration and Natural Logarithms*.)

After integrating both sides of the equation you get:

$$\ln(y) = \ln(x^2 + 1) + c$$

where the constant *c* results from adding together the constants introduced from integrating each side of the equation. Using the logarithmic transformation, you get:

$$e^{\ln(y)} = e^{\ln(x^2+1)+1} = e^{\ln(x^2+1)} e^c = Ae^{\ln(x^2+1)} \qquad \text{(where } A = e^c\text{)}$$

which becomes

$$y = A(x^2 + 1) \qquad \text{which is the } \textbf{general solution} \text{ to } (x^2+1)\frac{dy}{dx} = 2xy.$$

3.

a. In the solution to question 4 part (a) it was shown that the **general solution** to

$$2y\frac{dy}{dx} = 4(x^2 + 1) \qquad \text{is} \qquad y = \sqrt{\frac{4}{3}x^2 + 4x + k}$$

To find the **particular solution** you need to use the initial condition $y(0) = 2$.

As *y* is itself a function of *x*, you can write $y = y(x)$. The initial condition

$$y(0) = 2 \qquad \text{tells you that when} \qquad x = 0\text{, then } y = 2.$$

You can substitute this into your **general solution**:

$$y = \sqrt{\frac{4}{3}x^2 + 4x + k} \qquad \text{becomes} \qquad y(0) = 2 = \sqrt{0 + 0 + k} = \sqrt{k}$$

So $\sqrt{k} = 2$, and you can square both sides gives $k = 4$.

Substituting this into $y = \sqrt{\frac{4}{3}x^2 + 4x + k}$ gives:

$$y = \sqrt{\frac{4}{3}x^2 + 4x + 4}$$

which is the **particular solution** to $2y\frac{dy}{dx} = 4(x^2 + 1)$ when $y(0) = 2$.

b. In the solution to question 4 part (b) it was shown that the **general solution** to

$$\frac{dy}{dx} = y^2\cos(x) \qquad \text{is} \qquad y = -\frac{1}{\sin(x) + c}.$$

To find the **particular solution** you need to use the initial condition $y(0)=2$. As *y* is itself a function of *x*, you can write $y=y(x)$. The initial condition

$y(0)=2$ tells you that when $x=0$, then $y=2$.

You can substitute this into your **general solution**:

$$y=-\frac{1}{\sin(x)+c} \quad \text{becomes} \quad 2=-\frac{1}{c}$$

Then to find *c* you can multiply both sides of the equation by *c* and then divide both sides by 2 to give:

$$c=-\frac{1}{2}=-0.5$$

so $y=-\dfrac{1}{\sin(x)-0.5}$ is the **particular solution** to $\dfrac{dy}{dx}=y^2\cos(x)$ when $y(0)=2$.

c. In the solution to question 4 part (c) it was shown that the **general solution** to

$$(x^2+1)\frac{dy}{dx}=2xy \quad \text{is} \quad y=A(x^2+1).$$

To find the **particular solution** you need to use the initial condition $y(0)=2$. As *y* is itself a function of *x*, you can write $y=y(x)$. The initial condition

$y(0)=2$ tells you that when $x=0$, then $y=2$.

You can substitute this into your **general solution**:

$$y=A(x^2+1) \quad \text{becomes} \quad 2=A(0+1) \quad (\text{so } A=2),$$

So $y=2(x^2+1)$ is the **particular solution** to $(x^2+1)\dfrac{dy}{dx}=2xy$ when $y(0)=2$.