# Learning Enhancement Team

University of East Anglia

STUDENT SUPPORT SERVICE

***Steps into Differential Equations***

# Separable Differential Equations

***This guide helps you to identify and solve separable first-order ordinary differential equations.***

## Introduction

A **differential equation** (or **DE**) is any equation which contains a function and its derivatives, see study guide: *Basics of Differential Equations*. To make the best use of this guide you will need to be familiar with some of the terms used to categorise differential equations.

**Linear DE**: The function *y* and any of its derivatives can **only** be multiplied by a constant or a function of *x*.

**Nonlinear DE**: More complicated functions of *y* and its derivatives appear **as well as** multiplication by a constant or a function of *x*.

**Ordinary differential equation (ODE)**: Contains ordinary derivatives.

**First order DE**: The highest order derivative is first order, $\frac{dy}{dx}$ or $y'$ or $\dot{x}$.

This guide is only concerned with first order ODEs and will help you develop strategies to solve them. A first order ODE is called **separable** if it can be written in general as:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$ **Separable first order ODE**

where $\frac{dy}{dx}$ is the first derivative of *y* with respect to *x*, $f(x)$ is a function of *x* and $g(y)$ is a function of *y*. The general form of the separable ODE can be rearranged so that **all the functions of *y* are on one side of the equals sign and all the functions of *x* are on the other side**. This is written as:

$$g(y)dy = f(x)dx$$ **Separable first order ODE with variables separated**

This important technique in mathematics is called **separation of variables**. If you have a separable first order ODE it is a good strategy to separate the variables. If you have any constants and/or coefficients it is a good strategy to include them as part of $f(x)$.

| **Top Tip**: | Include any constants as part of $f(x)$ |
|---|---|

You may have noticed that when separating variables, the derivative has been treated like a fraction. A Learning Enhancement Tutor will be happy to talk to you in more depth about this.

As you shall see, integration is a powerful tool for solving separable ordinary differential equations so being comfortable with the basics of integration is essential if you want to be able to solve these ODEs. You can use the resources: *Steps into Calculus* to help you with your integration and differentiation skills.

*Example*: Separate the variables in these differential equations, if possible.

(a) $\frac{dy}{dx} = 2x$ (b) $y\frac{dy}{dx} + 4 = 2x$ (c) $\frac{dy}{dx} = \frac{3x^2 + y^2}{xy}$

(d) $e^x \frac{dy}{dx} = \frac{1}{\cos(y)}$ (e) $\frac{dy}{dx} = 2y$ (f) $\frac{dy}{dx} + 4x^2 y = \sin(x)$

(a) You can multiply each side of this equation by $dx$ to give:

$$dy = 2xdx$$

Now the variables are separated with functions of $y$ on the left-hand side of the equals sign and functions of $x$, along with constant 2, on the right-hand side.

(b) You can subtract 4 from each side of the DE and then multiply by $dx$ to take the expression to the form $g(y)dy = f(x)dx$:

$$ydy = (2x - 4)dx$$

(c) You cannot separate the variables here. In fact this is a **homogeneous** type of differential equation and requires a special method to solve it (see study guide: *Homogeneous First Order Differential Equations*).

(d) You can multiply each side by $\cos y\, dx$ and $dx$, then divide by $e^x$ to give:

$$\cos(y)\, dy = e^{-x} dx$$

.

(e) You can divide each side of the equation by $y$ and then multiply by $dx$ such that:

$$\frac{1}{y}dy = 2dx$$

(f) You cannot separate the variables here. The solution of these types of DE is discussed in the study guide: *Linear First Order Differential Equations*.

## Solving separable first order ODEs

### 1. Most simple case: when $g(y)$ is a constant

In the separable differential equation on page 1 of this guide, if $g(y)$ is constant you get:

$$\frac{dy}{dx} = f(x) \qquad \text{or, after separating the variables} \qquad dy = f(x)dx$$

where any constants are part of $f(x)$.
If you look carefully, the equation on the left can be read as "the derivative of *y* with respect to *x* is equal to $f(x)$" which looks like the answer to a question in differentiation.

Solving this type of differential equation and finding *y* involves undoing the differentiation. In other words the question is "what function can be differentiated to give $f(x)$?". As indefinite integration is the inverse of differentiation, if you integrate $f(x)$ with respect to $x$ you will find the function *y*. So if:

$$\frac{dy}{dx} = f(x) \quad \text{then} \quad \int dy = \int f(x)dx \quad \text{which results in} \quad y = \int f(x)dx$$

You can only solve these types of problem if you can integrate $f(x)$. See study guide: *What is Integration?* for more details about integrating.

*Example*: Solve $\frac{dy}{dx} = 2x$.

This is question (a) from the second page of this guide and is asking you to find the function *y* which has a derivative with respect to *x* which is equal to $2x$.

You can find *y* by integrating $2x$:

$$y = \int 2x\,dx = x^2 + c \qquad \textbf{general solution} \text{ of } \frac{dy}{dx} = 2x$$

So $y = x^2 + c$ and **you can check your answer by differentiation** to give $\frac{dy}{dx} = 2x$.

However you now have a **constant** *c* in your solution. Solutions which contain these unknown constants or coefficients are called **general solutions**. So the general solution to this problem is $y = x^2 + c$. You are often given extra information about a differential equation, which is called a **boundary condition**. Boundary conditions are used to help you evaluate any unknown constants in your problem.

*Example*: Solve $\frac{dy}{dx} = 2x$ subject to the boundary condition $y(2) = 0$.

Remember that *y* is a function of *x*, which can be written as $y(x)$. So the boundary condition $y(2) = 0$ tells you that when $x = 2$, $y = 0$. When you substitute the boundary condition into the general solution (found in the previous example):

$y = x^2 + c$ becomes $0 = 2^2 + c$

or in other words, $c = -4$. You can now substitute this value back into your general solution to give a solution particular to the given boundary conditions:

$y = x^2 - 4$ **particular solution** of $\frac{dy}{dx} = 2x$ **when** $y(2) = 0$

A solution like this one, with no unknown constants, is called the **particular solution**. You can still check your answer using differentiation to see that your solution is valid.

## 2. General method for separable first order ODEs

Separable first order ODEs, defined by the general equations on the first page of this guide, are often the first type of DE you learn to solve. The first step towards solving them requires you to separate the variables, as shown on page 2. Once this is done, you can move on to the next stage which involves indefinite integration.

*Example*: Solve $y\frac{dy}{dx} + 4 = 2x$ where $y(0) = 0$.

Firstly separate the variables, covered in example (b) on page 2, to get:

$$ydy = (2x - 4)dx$$

This is the equation on page 1 with $g(y) = y$ and $f(x) = 2x - 4$. To find the general solution to a differential equation after separating the variables, you integrate both sides of the equation. So:

$$\int ydy = \int (2x - 4)dx$$

After doing the integration shown on both sides of the equals sign you get:

$\frac{y^2}{2} = x^2 - 4x + c$ **general solution** of $y\frac{dy}{dx} + 4 = 2x$

Where the constant $c$ results from collecting together the two integration constants resulting from the indefinite integration on both sides of the equation. Although this is the general solution to the problem, it can often be useful to rearrange it to give "$y = ...$", especially if you are going to check your answer using differentiation.

Now the boundary condition $y(0) = 0$ tells you that when $x = 0$, $y = 0$ which implies that $c = 0$ and gives the particular solution of:

$$\frac{y^2}{2} = x^2 - 4x \quad \textbf{particular solution} \text{ of} \qquad y\frac{dy}{dx} + 4 = 2x \quad \text{for} \quad y(0) = 0$$

*Example*: Solve $\dfrac{dy}{dx} = 2y$ .

This kind of differential equation is very common in science and economics as it describes behaviour where the rate of change of *y* is proportional to the variable *y* itself. In this case the rate of change of *y* with respect to *x* is equal to twice the value of *y*. Here you can separate the variables by dividing by *y* and multiplying by $dx$ to give:

$$\frac{1}{y}dy = 2dx$$

This is the equation on the first page of this guide with $g(y) = 1/y$ and $f(x) = 2$.

Next you integrate both sides of the equals sign which gives:

$$\int \frac{1}{y}dy = \int 2dx \qquad \text{integrate to get} \qquad \ln y = 2x + c$$

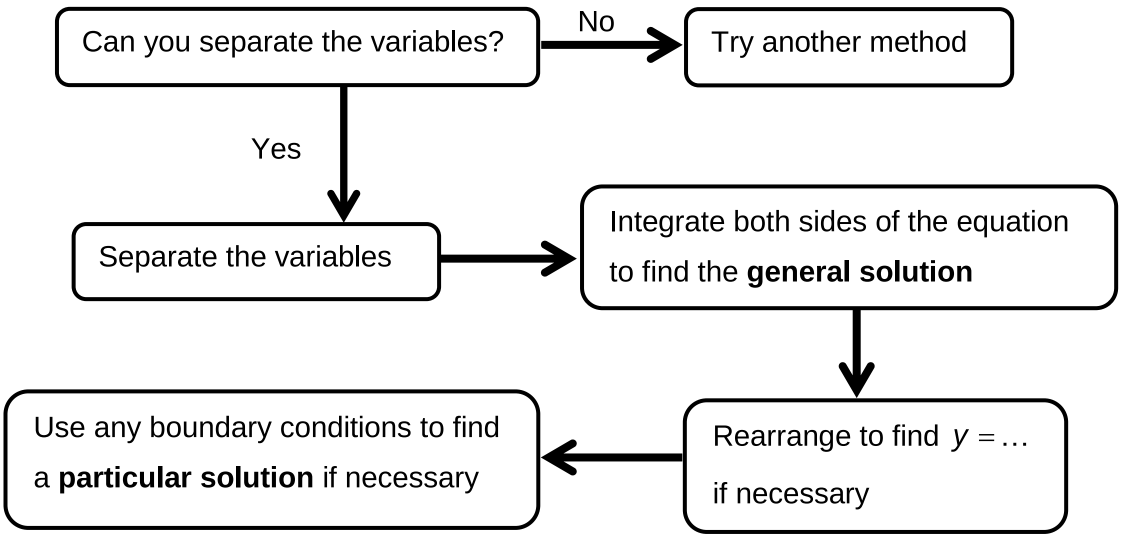
again the constant $c$ results from collecting together the two constants of integration resulting from the indefinite integration on both sides of the equation. The logarithmic transformation (see study guide: *Basics of Logarithms*) tells you that:

$$y = e^{2x+c} \qquad \text{or} \qquad y = Ae^{2x} \qquad (\text{where } A = e^c)$$

So the general solution to this problem is an exponential function (see study guide: *Exponential Functions*). This explains why so many problems and phenomena in science and economics (radioactive decay, first order rate kinetics, compound interest) exhibit exponential behaviour.

> In a system where the rate of change of a variable is directly proportional to the variable itself, the function describing the behaviour of the variable will be an exponential function.

## Flow chart for solving separable first order ODEs




## Want to know more?

If you have any further questions about this topic you can make an appointment to see a **Learning Enhancement Tutor** in the **Student Support Service**, as well as speaking to your lecturer or adviser.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| 💻 | Ask: | ask.let@uea.ac.uk |
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