Learning Enhancement Team

University of East Anglia
STUDENT SUPPORT SERVICE

Steps into Differential Equations

Basics of Differential Equations

This guide explains what a differential equation is and also describes the language used to categorise them. It also discusses the different kind of solutions to differential equations that you may come across.

Introduction

It is very common, especially in science, for a relationship you are considering to involve a **derivative** of some kind. The study guide: *What is Differentiation?* explains that a derivative is rate of change of one variable with respect to another. For example in physics, acceleration is the rate of change of velocity with respect to time. As a result of this, equations and relationships in science are often described by **differential equations** or simply "**DE's**". For example:

Rate Kinetics: $\dfrac{dC}{dt} = kC$ Acceleration: $a = \dfrac{d^2x}{dt^2}$

In the first equation the rate of change of the concentration C with respect to time t depends on C itself multiplied by a constant k. In the second equation the acceleration a equals the second derivative of the position x with respect to t.

A differential equation is a mathematical relationship between functions and their derivatives.

The derivatives found in differential equations can be written in many different forms:

Leibniz form:	$\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}$	and so on.
Lagrange form:	y', y''	and so on.
Newtonian form:	$\dot{x}, \ddot{x}$	and so on (time derivative only).

This study guide will help you identify the different types of DE's you may come across.

To **solve** a differential equation you need to find a function which fulfils the constraints of the equation. For example, in the rate kinetics equation above, the concentration C is described by a function whose derivative is equal to C itself multiplied by k. You can work out the nature of the functions involved in differential equations in many ways and, as DE's contain derivatives, it makes sense that their study follows on from basic calculus. It is very useful for you to have a good grasp of the language of and methods involved in **differentiation** and **integration** before beginning to study, and more importantly attempting to solve, DE's. You can explore the range of resources: *Steps into Calculus* which can help improve your basic calculus if necessary.

Classification of differential equations

Mathematicians are fond of categorising things and differential equations are no exception. In the rest of this guide you can assume that the function under scrutiny is *y* and *y* itself is a function of a variable *x*. (In the acceleration equation on page 1, the function of interest is the position and position itself is a function of time.)

Differential equations can be one of two fundamental types, either **linear** or **non-linear**.

Linear DE:	The function *y* and any of its derivatives can **only** be multiplied by a constant or a function of *x*.

Non-linear DE:	More complicated functions of *y* and its derivatives appear **as well as** multiplication by a constant or a function of *x*

As well as being linear or non-linear, a differential equation is also characterised by the type of derivatives it contains.

Ordinary differential equation (ODE):	Contains only ordinary derivatives

Partial differential equation (PDE):	Contains partial derivatives

Some of the most famous and important differential equations are PDE's.

The **order of the differential equation** is given by the highest order derivative in the equation. For example, the most common orders are :

First order DE:	Highest order derivative is $\frac{dy}{dx}$ or y' or $\dot{x}$

Second order DE:	Highest order derivative is $\frac{d^2y}{dx^2}$ or y'' or $\ddot{x}$

The order of the derivative in Lagrange notation is the number of primes (dashes) next to a variable and in Newtonian notation is the number of dots above a variable.

Example: Classify the following differential equations:

(i) $y'' + 3y' + 2y = 0$ *Linear second order ODE*

Linear as *y* and its derivatives are only multiplied by constants, second order due to the second derivative y'' and ordinary as only ordinary derivatives are present.

(ii) $\left(\dfrac{dy}{dx}\right)^2 + e^x y = 0$ *Non-linear first order ODE*

Non-linear due to the squaring of the y' term, first order due to the first derivative y' and ordinary as only ordinary derivatives are present.

Here, it is worth noting that $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d^2y}{dx^2} \neq \left(\dfrac{dy}{dx}\right)^2 = \dfrac{dy}{dx} \cdot \dfrac{dy}{dx}$.

So y'' in (i) and $(dy/dx)^2$ in (ii) are not the same.

(iii) $\dfrac{dy}{dx} + \cos y = 0$ *Non-linear first order ODE*

Non-linear due to the "$\cos y$" term, first order due to dy/dx and ordinary as only ordinary derivatives are present.

(iv) $xy'' - \cos x y' + 2x = 0$ *Linear second order ODE*

Linear DE as *y* and its derivatives are only multiplied by functions of *x* (even though these functions are non-linear, it is the functions of *y* which control the linear/non-linear nature of the DE), second order due to y'' and ordinary as only ordinary derivatives are present.

(v) $\dfrac{d^3y}{dx^3} + 3\dfrac{dy}{dx} + y = 0$ *Linear third order ODE*

Linear because *y* and its derivatives are only multiplied by constants, third order due to the third derivative d^3y/dx^3, and ordinary as only ordinary derivatives are present.

(vi) $(y'')^3 + 3y = \sin x$ *Non-linear second order ODE*

Non-linear because y'' is cubed, second order due to the second derivative y'' and ordinary as only ordinary derivatives are present.

(vii) $\dfrac{\partial u}{\partial t} = k^2 \dfrac{\partial^2 u}{\partial x^2}$ (*k* is a constant) *Linear second order PDE*

This is the famous **one-dimensional heat equation** which is linear because the derivatives are only multiplied by constants, partial as partial derivatives are present and second order due to $\partial^2 u / \partial x^2$.

Solving differential equations

Some of the greatest mathematicians who ever lived have devised methods to solve specific types of differential equations and often you will see equations which are named after the person who solved them.

$$\left(1-x^2\right)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+n(n+1)y=0$$

Legendre's equation

2nd order linear ODE: common in problems with spherical symmetry

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-n^2\right)y=0$$

Bessel's equation

2nd order linear ODE: common in problems with cylindrical symmetry

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

Three-dimensional Laplace equation

2nd order linear PDE common across various disciplines in physics

There are hundreds more whose solutions require knowledge of a wide-range of brilliant techniques. **You can greatly improve your mathematical skill by reading about how various differential equations were solved.**

There are many differential equations which can be solved by mathematical methods to give an exact function which satisfies the DE and these tend to be those which fit a specific form or pattern (see study guides: *Separable Differential Equations*, *Linear First Order Differential Equations*, *Homogeneous Differential Equations* and *Second Order Ordinary Differential Equations*). If a DE can be solved to give an **explicit** or **implicit function** then the solution is said to be **analytical**.

Many different types of differential equations **cannot be solved by mathematical means** to give an analytical solution. Commonly, computer models are used to find out how these kinds of DE's behave. For example, weather forecasts are determined this way. Such "solutions" are not really solutions at all but **approximations** to the exact solution. However these approximations are often referred to as **numerical solutions** as numbers are inputted directly into the differential equation to determine the output.

Solving differential equations: general solutions

Solving a differential equation finds functions which satisfy the conditions described by the DE. For example the linear second order ODE:

$$y'' + 3y' + 2y = 0$$

describes a function y where the sum of its second derivative y'' plus three times its first derivative $3y'$ plus twice itself $2y$ is equal to zero. Solving this ODE means the same as finding any functions which satisfy this relationship.

Example: Show that both $y = e^{-2x}$ and $y = e^{-x}$ satisfy $y'' + 3y' + 2y = 0$.

Begin by differentiating: $y = e^{-2x}$, $y' = -2e^{-2x}$ and $y'' = 4e^{-2x}$

Then substitute these results into your differential equation:

$y'' + 3y' + 2y = 0$ becomes $4e^{-2x} - 6e^{-2x} + 2e^{-2x} = 0$ which is true.

So $y = e^{-2x}$ is a function which "solves" this ODE. Now look at the function $y = e^{-x}$.

Again, begin by differentiating: $y = e^{-x}$, $y' = -e^{-x}$ and $y'' = e^{-x}$

Then substitute these new results into your differential equation:

$y'' + 3y' + 2y = 0$ becomes $e^{-x} - 3e^{-x} + 2e^{-x} = 0$ which again is true.

So $y = e^{-x}$ is also a function which "solves" this ODE. You can also multiply either of these functions by a constant and *still* get a solution to the ODE. It seems like you have solutions which look like either $y = Ae^{-2x}$ or $y = Be^{-x}$ where A and B are **constants** (see discussion below). Both of these solutions give zero when substituted into the ODE and so you can add them together and *still* get a solution as zero plus zero is zero. So the solution looks like:

$y = Ae^{-2x} + Be^{-x}$ **general solution** of $y'' + 3y' + 2y = 0$

All of methods of solving DE's introduce **unknown constants** (usually simply called constants) and they often appear as letters in alphabetical order. There are various ways constants are introduced into DE's. For example, as integration is the inverse of differentiation, you would correctly expect that indefinite integration is often used to find the solution of a DE. Using indefinite integration introduces a constant into your answer as "$+c$". Solutions to DE's which contain unknown constants or coefficients, such as the one above, are called **general solutions**.

The **number of different unknown constants** in a **general solution** of a differential equation is equal to the **order** of that differential equation.

Solving differential equations: particular solutions (using initial conditions and boundary conditions)

If more is known about the situation that a differential equation describes, you may have extra information available which enables you to calculate the constants in your general solution. These conditions are called **initial** or **boundary conditions** and are given as mathematical statements relating to the DE you are solving. Using boundary conditions is covered in depth in the study guides for specific types of differential equation, see *Steps Into Differential Equations* for more details. However, an example of boundary conditions for $y'' + 3y' + 2y = 0$ may be:

$y(0) = 1$	which tells you that when	$x = 0$, $y = 1$
$y'(0) = 1$	which tells you that when	$x = 0$, $y' = 1$

You can substitute these into the general solution to get simultaneous equations which imply that $A = -2$ and $B = 3$. When you have found any constants, you substitute them into the general solution of your DE. You should now have a solution which does not contain unknown constants and this is called a **particular solution** and so:

$y = -2e^{-2x} + 3e^{-x}$ is the **particular solution** of $y'' + 3y' + 2y = 0$ for $y(0) = 1$, $y'(0) = 1$

Want to know more?

If you have any further questions about this topic you can make an appointment to see a **Learning Enhancement Tutor** in the **Student Support Service**, as well as speaking to your lecturer or adviser.

Call: 01603 592761

Ask: ask.let@uea.ac.uk

Click: https://portal.uea.ac.uk/student-support-service/learning-enhancement

There are many other resources to help you with your studies on our website.
For this topic there is a webcast.

Your comments or suggestions about our resources are very welcome.

Scan the QR-code with a smartphone app for a webcast of this study guide.