University of East Anglia
STUDENT SUPPORT SERVICE

# Integration by Parts

***This guide defines the formula for integration by parts. It gives advice about when to use the integration by parts formula and describes methods to help you use it effectively.***

## Introduction

Integration and differentiation are the two parts of calculus and, whilst there are well-defined rules to differentiate almost all functions, being able to integrate any given function is not so easy and requires experience as there are many techniques. Although integrating basic functions is a fundamental skill (see study guide: *Integrating Basic Functions*), often you will find that the function you have been given to integrate is more complicated. One common method of integrating more complicated functions of a certain type is **integration by parts**. As with differentiating, it is crucial when integrating to be able to identify the form of the function you have to integrate. Once you know the form of the function you can make a good decision about the rule you wish to employ. **You can use the method of integration by parts to integrate many complicated functions which are made by multiplying together more basic functions**. You can learn more about these types of functions in the study guides: *More Complicated Functions* and *The Product Rule*. If you are unfamiliar with the language used to describe integration and differentiation, you may find it useful to read the study guides: *What is Differentiation?*, *What is Integration?* and *Definite Integrals* before carrying on with this guide.

The integration by parts formula for indefinite integrals is given by:

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

On the **left-hand side** of the formula you have an **integrand** which may look strange because it made by the product of a function $u$ and the derivative of a function $v$, $dv/dx$. Both $u$ and $v$ are functions of $x$. **You use the method of integration by parts to integrate more complicated functions which are the product of two basic functions**. However, unlike the product rule from which integration by parts is derived, the substitutions that you make are not $u$ and $v$ but $u$ and $dv/dx$. Take the following integral:

$$\int 2x\cos x\,dx$$

As the integrand, $2x\cos x$, is a product of the two basic functions $2x$ and $\cos x$, it is appropriate to try integration by parts. You use the integration by parts formula by first assigning $u$ to one of these functions in the integrand and $dv/dx$ to the other. **Usually, if a power function is part of the integrand it is good practice to let it equal *u*,** the choice of $dv/dx$ is the other function by default. The reason for this is that you differentiate $u$ and so reduce the power of $x$ by 1, this leads to a simpler integral on the right hand side of the integration by parts formula. (A table listing common functions and good choices of $u$ and $dv/dx$ is found at the end of this guide.) In this example, which contains a power function, it is sensible to make $u = 2x$ which means that $dv/dx = \cos x$.

Next, to solve the integral, you need to find $v$ and $du/dx$ and substitute them, along with $u$, into the right hand side of the integration by parts formula. Remember you have to:

**Differentiate** $u$ to find $\dfrac{du}{dx}$

**Integrate** $\dfrac{dv}{dx}$ to find $v$

Integration by parts is so named, because you differentiate the one 'part' of the integrand ($u$) and integrate the other 'part' ($dv/dx$). Therefore it is important that you know how to differentiate and integrate basic functions to use integration by parts effectively (see study guides: *Differentiating Basic Functions* and *Integrating Basic Functions*). Here $du/dx = 2$ and $v = \sin x$. Substitute these results, along with $u = 2x$, into the right-hand side of the integration by parts formula to give:

$$uv = 2x\sin x \qquad \text{and} \qquad \int v\frac{du}{dx}dx = \int 2\sin x\, dx.$$

The right hand side of the rule gives the "answer" to the integral. It is also unusual as it contains two terms, the second of which is another integral that you must perform by an appropriate method in order to find the answer. If you assign $u$ and $dv/dx$ to the correct functions in the left hand side of the formula, then the integration on the right-hand side should be either **as simple as** or **simpler than the original problem**. If you do not, the integration will be more difficult than the original problem and you should swap your substitutions around. Here:

$$\int 2\sin x\, dx \qquad \text{is simpler than} \qquad \int 2x\cos x\, dx$$

and so the choice of $u = 2x$ and $dv/dx = \cos x$ is good.

If you are unsure whether to use integrations by parts, or another integration method, the most important question to ask yourself is 'Am I integrating a product function?'. Practicing problems is an excellent way of gaining experience of integration methods as it helps you make good decisions about which method to use for certain types of problems.

When you are answering longer, more complicated problems, you may find it useful to divide your page vertically into an **answer space** (where your answer will be written) and an **exploring space** (where you can do any working and thinking which informs your answer).

*Answer Space*

What is $\int 2x\cos x\,dx$

*Use integration by parts*

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int 2x\cos x\,dx = 2x\sin x - \int (\sin x)\cdot 2dx$$

$$\int 2x\cos x\,dx = 2x\sin x - (-2\cos x) + c$$

$$\int 2x\cos x\,dx = 2x\sin x + 2\cos x + c$$

*Exploring Space*

Let $u = 2x$ as it is a power function, so $\frac{dv}{dx} = \cos x$

For the formula you need $u$, $du/dx$ and $v$:

$$\frac{du}{dx} = 2 \qquad \text{and} \qquad v = \int \cos x\,dx = \sin x + c$$

Arrows can help with the substitution.

The indefinite integral on the right hand side is:

$$\int (\sin x)\cdot 2dx = \int 2\sin x\,dx = -2\cos x + c$$

Using brackets can help with the minus signs.

Open the brackets to find the answer.

**Check you are correct by differentiating your answer using the product rule**, this should give your original integrand.

## Calculating definite integrals using integration by parts

The study guide: *Definite Integrals*, describes how definite integrals can be used to find the areas on graphs. The integration by parts formula for definite integrals is analogous to the formula for indefinite integrals. So, between the limits $x = a$ and $x = b$:

$$\int_a^b u\frac{dv}{dx}dx = [uv]_a^b - \int_a^b v\frac{du}{dx}dx$$

*Example*: What is $\int_1^4 2x\cdot e^{3x}dx$

*Answer Space*

What is $\int_1^4 2x\cdot e^{3x}dx$

*Use integration by parts (definite)*

$$\int_a^b u\frac{dv}{dx}dx = [uv]_a^b - \int_a^b v\frac{du}{dx}dx$$

$$\int_1^4 2xe^{3x}dx = \left[2x\cdot\tfrac{1}{3}e^{3x}\right]_1^4 - \int_1^4 \tfrac{1}{3}e^{3x}\cdot 2dx$$

*Exploring Space*

Let $u = 2x$ as it is a power function, so $\frac{dv}{dx} = e^{3x}$

For the formula you need $u$, $du/dx$ and $v$:

$$\frac{du}{dx} = 2 \qquad \text{and} \qquad v = \int e^{3x}dx = \tfrac{1}{3}e^{3x}$$

The definite integral on the right hand side is:

$$\int_1^4 \tfrac{1}{3}e^{3x}\cdot 2dx = \int_1^4 \tfrac{2}{3}e^{3x}dx = \left[\tfrac{2}{9}e^{3x}\right]_1^4$$

$$\int_1^4 2xe^{3x}dx = \left[\tfrac{2}{3}xe^{3x}\right]_1^4 - \left[\tfrac{2}{9}e^{3x}\right]_1^4$$

Now follow the method for working out definite integrals given in the study guide: *Definite Integrals*

$$\int_1^4 2xe^{3x}dx = \left(\tfrac{8}{3}e^{12} - \tfrac{2}{3}e^{3}\right) - \left(\tfrac{2}{9}e^{12} - \tfrac{2}{9}e^{3}\right)$$

Take care with minus signs, using brackets can help

$$\int_1^4 2xe^{3x}dx = \tfrac{22}{9}e^{12} - \tfrac{4}{9}e^{3}$$

This is the answer

## Using integration by parts more than once

When integrals contain higher powers of *x*, you may need to use integration by parts more than once to obtain a solution. In general, if an integral contains $x^n$ then you will need to use integration by parts *n* times.

*Example*: What is $\int x^2 \sin x\, dx$ ?

| *Answer Space* | *Exploring Space* |
|---|---|
| What is $\int x^2 \sin x\, dx$ | Let $u = x^2$ as it is a power function, so $\frac{dv}{dx} = \sin x$ |
| *Use integration by parts (indefinite)* | |
| $\int u \frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ | For the formula you need $u$, $du/dx$ and $v$ : |
| | $\frac{du}{dx} = 2x$ and $v = \int \sin x dx = -\cos x + c$ |
| $\int x^2 \sin x dx = x^2(-\cos x) - \int(-\cos x)2x dx$ | Arrows can help with the substitution. |
| $\int x^2 \sin x dx = -x^2\cos x + \int 2x\cos x dx$ | Be careful with the minus signs. |

Notice that the integral that is part of the answer is less complex than the original problem. This is a good sign. In fact this integral is the same as the one in the first example in this study guide so, using the answer from that example:

$$\begin{aligned}\int x^2 \sin x dx &= -x^2\cos x + \int 2x\cos x dx \\ &= -x^2\cos x + 2x\sin x + 2\cos x + c\end{aligned}$$

which you can check by differentiation using the product rule. As you used integration by parts in the first example, you have had to use the method twice to find this answer and the power of *x* in the question was 2.

## Solving recurrent integrals using integration by parts

There are some types of problems, those which involve an exponential function multiplied by a trigonometric function, where using integration by parts will not generate a simpler integral in the answer but instead will produce an integral of the same level of complexity as in the question. As neither of the parts of this product decrease in complexity when they are differentiated, using integration by parts on these problems has a circular nature. Solving these problems is an excellent example of elegance in mathematics as the method regenerates the original integral which allows a solution to be obtained by rearrangement.

*Example*: Determine $I = \int e^x \cos x \, dx$.

| *Answer Space* | *Exploring Space* |
|---|---|
| Integrate $\int e^x \cos x \, dx$ | Let $u = e^x$ and $\frac{dv}{dx} = \cos x$ |
| *Use integration by parts (indefinite)* | |
| $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$ | For the formula you need $u$, $du/dx$ and $v$: |
| | $\frac{du}{dx} = e^x$ and $v = \int \cos x dx = \sin x + c$ |
| $I = \int e^x \cos x dx = e^x \sin x - \int \sin x \cdot e^x dx$ | |

Notice that the integral you have generated is similar to the original question with $\cos x$ replaced by $\sin x$. It seems sensible to perform this integration by parts as well.

| *Answer Space* | *Exploring Space* |
|---|---|
| Integrate $\int e^x \sin x \, dx$ | Let $u = e^x$ and $\frac{dv}{dx} = \cos x$ |
| *Use integration by parts (indefinite)* | |
| $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$ | For the formula you need $u$, $du/dx$ and $v$: |
| | $\frac{du}{dx} = e^x$ and $v = \int \sin x dx = -\cos x + c$ |
| $\int e^x \sin x dx = -e^x \cos x + \int e^x \cos x dx$ | |

Substituting this result into the first part of the answer gives:

$$\int e^x \cos x dx = e^x \sin x + e^x \cos x - \int e^x \cos x dx$$

You can see that both the left- and right-hand sides of this equation contains the original question $I$. So, re-expressing the result in terms of $I$ you get:

$$I = e^x \sin x + e^x \cos x - I$$

So, after rearranging for $I$ by adding $I$ to both sides and then dividing by 2, you get:

$$I = \int e^x \cos x dx = \frac{e^x \sin x + e^x \cos x}{2} + c$$

## When to and when not to use integration by parts

The following table gives some general product functions which can be integrated using integration by parts and suggestions of what to choose for $u$ and $dv/dx$.

| General form ($a$, $b$, $n$ constants) | For $u$ choose | For $dv/dx$ choose |
|---|---|---|
| $\int ax^n \sin(bx)dx$ | $ax^n$ | $\sin(bx)$ |
| $\int ax^n \cos(bx)dx$ | $ax^n$ | $\cos(bx)$ |
| $\int ax^n e^{bx} dx$ | $ax^n$ | $e^{bx}$ |
| $\int ax^n \ln(bx)dx$ | $\ln(bx)$ | $ax^n$ |
| $\int \ln x\, dx$ | $\ln x$ | 1 |

Integration by parts is not always the easiest method to use, especially with trigonometric integrands. Take $\sin^2 x$, which can be written as a $\sin x \cdot \sin x$, integration by parts can be used but it is much simpler to use trigonometric identities and integrate $\frac{1}{2} - \frac{1}{2}\cos 2x$ instead.

## Want to know more?

If you have any further questions about this topic you can make an appointment to see a **Learning Enhancement Tutor** in the **Student Support Service**, as well as speaking to your lecturer or adviser.

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- Ask: ask.let@uea.ac.uk
- Click: https://portal.uea.ac.uk/student-support-service/learning-enhancement

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