Learning Enhancement Team

University of East Anglia
STUDENT SUPPORT SERVICE

Model Answers: Integration by Parts

These are the model answers for the worksheet that has questions on integration by parts.

Integration by Parts study guide

a. $\int x\sin(x)dx$ is suitable for integration by parts with $u = x$ and $\frac{dv}{dx} = \sin(x)$.

b. $\int_{\pi}^{2\pi} \frac{1}{2}\cos(x)dx$ is a basic function (see study guide: *Integration of Basic Functions*).

c. $\int -\frac{x}{2}\cos(2x)dx$ is suitable for integration by parts with $u = -\frac{x}{2} = -\tfrac{1}{2}x$ and $\frac{dv}{dx} = \cos(2x)$.

d. $\int xe^{-x}dx$ is suitable for integration by parts with $u = x$ and $\frac{dv}{dx} = e^{-x}$.

e. $\int x\ln(x)dx$ is suitable for integration by parts with $u = \ln(x)$ and $\frac{dv}{dx} = x$.

f. $\int_1^2 \frac{x}{3}\sin(x)dx$ is suitable for integration by parts with $u = \frac{x}{3} = \tfrac{1}{3}x$ and $\frac{dv}{dx} = \sin(x)$.

g. $\int_{\pi}^{2\pi} \frac{1}{-2}\cos(x)dx$ is a basic function (see study guide: *Integration of Basic Functions*).

h. $\int_1^5 xe^{x}dx$ is suitable for integration by parts with $u = x$ and $\frac{dv}{dx} = e^{x}$

i. $\int_e^{2e} \ln(3x)dx$ is suitable for integration by parts with $u = \ln(3x)$ and $\frac{dv}{dx} = 1$.

j. $\int -\frac{1}{4}x^3 dx$ can be integrated using the power rule (see study guide: *Integrating using the Power Rule*).

k. $\int \frac{1}{2}\sin\left(\frac{x}{2}\right)dx$ is a basic function (see study guide: *Integration of Basic Functions*).

l. $\int_0^{\pi} x^2 \cos(x)dx$ is suitable for integration by parts with $u = x^2$ and $\frac{dv}{dx} = \cos(x)$.

m. $\int e^{3x}\sin(x)dx$ is suitable for integration by parts with $u = e^{3x}$ and $\frac{dv}{dx} = \sin(x)$.

n. $\int (x)^0 \cos(4x)dx$ remember that anything to the power zero is 1 and so this is a basic function (see study guide: *Integration of Basic Functions*).

o. $\int x^2 \ln(4x)dx$ is suitable for integration by parts with $u = \ln(4x)$ and $\frac{dv}{dx} = x^2$

2.

a. $\int x\sin(x)dx$ is suitable for integration by parts with $u = x$ and $\frac{dv}{dx} = \sin(x)$

Answer Space	*Exploring Space*
What is $\int x\sin(x)dx$	Let $u = x$ as it is a power function, so $\frac{dv}{dx} = \sin(x)$
Use integration by parts	
$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$	For the formula you need u, du/dx and v:
	$\frac{du}{dx} = 1$ and $v = \int \sin(x)dx = -\cos(x) + c$
$\int x\sin(x)dx = -x\cos(x) - \int -\cos(x)\cdot 1dx$	Arrows can help with the substitution.
	The indefinite integral on the right hand side is: $\int -\cos(x)\cdot 1dx = -\int \cos(x)dx = -\sin(x) + c$
$\int x\sin(x)dx = -x\cos(x) - (-\sin(x)) + c$	Using brackets can help with the minus signs.
$\int x\sin(x)dx = -x\cos(x) + \sin(x) + c$	Open the brackets to find the answer.

Check you are correct by differentiating your answer using the product rule, this should give your original integrand.

c. $\int -\frac{x}{2}\cos(2x)dx$ is suitable for integration by parts with $u=-\frac{1}{2}x$ and $\frac{dv}{dx}=\cos(2x)$.

Answer Space

What is $\int -\frac{x}{2}\cos(2x)dx$

Use integration by parts

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int -\frac{x}{2}\cos(2x)dx = -\frac{1}{2}x\cdot\frac{1}{2}\sin(2x) - \int \frac{1}{2}\sin(2x)\cdot\left(-\frac{1}{2}\right)dx$$

$$\int -\frac{x}{2}\cos(2x)dx = -\frac{1}{4}x\sin(2x) - \frac{1}{8}\cos(2x) + c$$

Exploring Space

Let $u=-\frac{1}{2}x$ as it is a power function, so

$$\frac{dv}{dx}=\cos(2x)$$

For the formula you need u, du/dx and v:

$$\frac{du}{dx}=-\frac{1}{2}$$

and $v=\int \cos(2x)dx = \frac{1}{2}\sin(2x)+c$

Arrows can help here.

The indefinite integral on the right hand side is:

$$\int \frac{1}{2}\sin(2x)\cdot\left(-\frac{1}{2}\right)dx = \int -\frac{1}{4}\sin(2x)dx$$

$$= \frac{1}{8}\cos(2x)+c$$

Check you are correct by differentiating your answer using the product rule, this should give your original integrand.

d. $\int xe^{-x}dx$ is suitable for integration by parts with $u = x$ and $\frac{dv}{dx} = e^{-x}$.

Answer Space

What is $\int xe^{-x}dx$

Use integration by parts

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int xe^{-x}dx = x\cdot\left(-e^{-x}\right) - \int\left(-e^{-x}\right)\cdot 1dx$$

$$\int xe^{-x}dx = x\cdot\left(-e^{-x}\right) - \left(e^{-x}\right) + c$$

$$\int xe^{-x}dx = -x\cdot e^{-x} - e^{-x} + c$$

Exploring Space

Let $u = x$ as it is a power function, so $\frac{dv}{dx} = e^{-x}$

For the formula you need u, du/dx and v:

$\frac{du}{dx} = 1$ and $v = \int e^{-x}dx = -e^{-x} + c$

Arrows can help with the substitution.
The indefinite integral on the right hand side is:

$$\int\left(-e^{-x}\right)\cdot 1dx = \int -e^{-x}dx = e^{-x} + c$$

Using brackets can help with the minus signs.

Open the brackets to find the answer.

Check you are correct by differentiating your answer using the product rule, this should give your original integrand.

e. $\int x\ln(x)dx$ is suitable for integration by parts with $u = \ln(x)$ and $\frac{dv}{dx} = x$.

Answer Space

What is $\int x\ln(x)dx$

Use integration by parts

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int x\ln(x)dx = \ln(x)\cdot\frac{x^2}{2} - \int\left(\frac{x^2}{2}\right)\cdot\frac{1}{x}dx$$

$$\int x\ln(x)dx = \ln(x)\cdot\frac{x^2}{2} - \left(\frac{x^2}{4}\right) + c$$

$$\int x\ln(x)dx = \ln(x)\cdot\frac{x^2}{2} - \frac{x^2}{4} + c$$

Exploring Space

Let $u = \ln(x)$, so $\frac{dv}{dx} = x$

For the formula you need u, du/dx and v :

$\frac{du}{dx} = \frac{1}{x}$ and $v = \int xdx = \frac{x^2}{2} + c$

Arrows can help with the substitution.

The indefinite integral on the right hand side is:

$$\int\left(\frac{x^2}{2}\right)\cdot\frac{1}{x}dx = \int\frac{x}{2}dx = \int\frac{1}{2}xdx = \frac{x^2}{4} + c$$

Using brackets can help with the minus signs.

Open the brackets to find the answer.

Check you are correct by differentiating your answer using the product rule, this should give your original integrand.

f. $\int_1^2 \frac{x}{3}\sin(x)dx$ is suitable for integration by parts with $u = \frac{x}{3} = \frac{1}{3}x$ and $\frac{dv}{dx} = \sin(x)$.

Answer Space

What is $\int_1^2 \frac{x}{3}\sin(x)dx$

Use integration by parts (definite)

$$\int_a^b u\frac{dv}{dx}dx = [uv]_a^b - \int_a^b v\frac{du}{dx}dx$$

$$\int_1^2 \frac{x}{3}\sin(x)dx = \left[\tfrac{1}{3}x\cdot(-\cos(x))\right]_1^2 - \int_1^2 (-\cos(x))\tfrac{1}{3}dx$$

$$= \left[-\tfrac{1}{3}x\cdot\cos(x)\right]_1^2 - \left[\tfrac{1}{3}\sin(x)\right]_1^2$$

Exploring Space

Let $u = \frac{1}{3}x$ as it is a power function, so $\frac{dv}{dx} = \sin(x)$.

For the formula you need u, du/dx and v:

$\frac{du}{dx} = \frac{1}{3}$ and $v = \int \sin(x)dx = -\cos(x)$

The definite integral on the right hand side is:

$$\int_1^2 -(\cos(x))\tfrac{1}{3}dx = \int_1^2 -\tfrac{1}{3}\cos(x)dx$$

$$= \left[\tfrac{1}{3}\sin(x)\right]_1^2$$

Now follow the method for working out definite integrals given in the study guide: *Definite Integrals*. So:

$$\left[-\tfrac{1}{3}x\cdot\cos(x)\right]_1^2 - \left[\tfrac{1}{3}\sin(x)\right]_1^2$$

$$= \left(-\tfrac{1}{3}2\cdot\cos(2) - \left(-\tfrac{1}{3}1\cdot\cos(1)\right)\right) - \left(\left(\tfrac{1}{3}\sin(2) - \tfrac{1}{3}\sin(1)\right)\right)$$

Take care with minus signs, using brackets can help

$$= -\tfrac{2}{3}\cdot\cos(2) + \tfrac{1}{3}\cdot\cos(1) + \tfrac{1}{3}\sin(2) - \tfrac{1}{3}\sin(1)$$

Do the calculation in radians

$$= 0.48$$

This is the answer in to 2 d.p

h. $\int_1^5 xe^x dx$ is suitable for integration by parts with $u = x$ and $\frac{dv}{dx} = e^x$.

Answer Space	*Exploring Space*
What is $\int_1^5 xe^x dx$	Let $u = x$ as it is a power function, so $\frac{dv}{dx} = e^x$
Use integration by parts (definite)	
$\int_a^b u\frac{dv}{dx}dx = [uv]_a^b - \int_a^b v\frac{du}{dx}dx$	For the formula you need u, du/dx and v:
$\int_1^5 xe^x dx = [x \cdot e^x]_1^5 - \int_1^5 (e^x) \cdot 1dx$	$\frac{du}{dx} = 1$ and $v = \int e^x dx = e^x$
$= [x \cdot (e^x)]_1^5 - [e^x]_1^5$	The definite integral on the right hand side is: $\int_1^5 (e^x) \cdot 1dx = \int_1^5 e^x dx = [e^x]_1^2$

Now follow the method for working out definite integrals given in the study guide: *Definite Integrals*. So:

$\int_1^5 xe^x dx = (5 \cdot (e^5) - (1 \cdot (e^1))) - (e^5 - e^1)$	Take care with minus signs, using brackets can help
$= 5e^5 - e - e^5 + e = 4e^5 = 593.65$	This is the answer to 2 d.p.

i. $\int_{e}^{2e} \ln(3x)dx$ is suitable for integration by parts with $u = \ln(3x)$ and $\frac{dv}{dx} = 1$.

Answer Space

What is $\int_{e}^{2e} \ln(3x)dx$

Use integration by parts (definite)

$$\int_a^b u\frac{dv}{dx}dx = [uv]_a^b - \int_a^b v\frac{du}{dx}dx$$

$$\int_e^{2e} \ln(3x)\cdot 1dx = [\ln(3x)x]_e^{2e} - \int_e^{2e} (x)\cdot\frac{1}{x}dx$$

$$= [x\ln(3x)]_e^{2e} - [x]_e^{2e}$$

Exploring Space

Let $u = \ln(3x)$, so $\frac{dv}{dx} = 1$

For the formula you need u, du/dx and v:

$\frac{du}{dx} = \frac{1}{x}$ and $v = \int 1dx = x$

The definite integral on the right hand side is:

$$\int_e^{2e} (x)\cdot\frac{1}{x}dx = \int_e^{2e} 1dx = [x]_e^{2e}$$

Now follow the method for working out definite integrals given in the study guide: *Definite Integrals*. So:

$[x\ln(3x)]_e^{2e} - [x]_e^{2e}$

$= (\ln(6e)2e - (\ln(3e)e)) - (2e - e)$ Take care with minus signs, using brackets can help

$= 2e\ln(6) + 2e - e\ln(3) - e - 2e + e$

$= e\ln(12) = 6.75$ This is the answer to 2 d.p.

I. $\int_0^{\pi} x^2 \cos(x)dx$ is suitable for integration by parts with $u = x^2$ and $\frac{dv}{dx} = \cos(x)$.

Answer Space

What is $\int_0^{\pi} x^2 \cos(x)dx$

Use integration by parts (definite)

$$\int_a^b u\frac{dv}{dx}dx = [uv]_a^b - \int_a^b v\frac{du}{dx}dx$$

$$\int_0^{\pi} x^2 \cos(x)dx = \left[x^2 \sin(x)\right]_0^{\pi} - \int_0^{\pi}(\sin(x))2x\,dx$$

Exploring Space

Let $u = x^2$ as it is a power function, so $\frac{dv}{dx} = \cos(x)$

For the formula you need u, du/dx and v:

$\frac{du}{dx} = 2x$ and $v = \int \cos(x)dx = \sin(x)$

The definite integral on the right hand side is:

$$\int_0^{\pi}\sin(x)2x\,dx = \int_0^{\pi}2x\sin(x)dx = 2\int_0^{\pi}x\sin(x)dx$$

You have already done this integration part a.:

$$\int x\sin(x)dx = -x\cos(x) + \sin(x) + c$$

And so:

$$2\int_0^{\pi}x\sin(x)dx = 2\left[-x\cdot\cos(x) + \sin(x)\right]_0^{\pi} = 2\pi$$

You can use this result in your answer:

$$\int_0^{\pi} x^2\cos(x)dx = \left[x^2\sin(x)\right]_0^{\pi} - 2\pi$$

$$= \left(\pi^2 \cdot (\sin(\pi)) - 0\right) - 2\pi = -2\pi$$

m. $\int e^{3x}\sin(x)dx$ is suitable for integration by parts with $u = e^{3x}$ and $\frac{dv}{dx} = \sin(x)$.

Answer Space

Integrate $\int e^{3x}\sin(x)dx$

Use integration by parts (indefinite)

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int e^{3x}\sin(x)dx = e^{3x}(-\cos(x)) - \int(-\cos(x))\cdot 3e^{3x}dx$$

Exploring Space

Let $u = e^{3x}$ and $\frac{dv}{dx} = \sin(x)$

For the formula you need u, du/dx and :

$\frac{du}{dx} = 3e^{3x}$ and $v = \int\sin(x)dx = -\cos(x) + c$

Notice that the integral you have generated is similar to the original question with $\sin x$ replaced by $-3\cos(x)$. It seems sensible to perform this integration by parts as well.

Answer Space

Integrate $\int -3\cos(x)e^{3x}dx$

Use integration by parts (indefinite)

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

Exploring Space

Let $u = -3e^{3x}$ and $\frac{dv}{dx} = \cos(x)$

For the formula you need u, du/dx and v:

$\frac{du}{dx} = -9e^{3x}$ and $v = \int\cos(x)dx = \sin(x) + c$

$$\int -3\cos(x)e^{3x}dx = -3e^{3x}(\sin(x)) - \int\sin(x)(-9e^{3x})dx$$

$$= -3e^{3x}(\sin(x)) + 9\int e^{3x}\sin(x)dx$$

Substituting this result into the first part of the answer gives:

$$\int e^{3x}\sin(x)dx = e^{3x}(-\cos(x)) - \int(-\cos(x))\cdot 3e^{3x}dx$$

$$= -e^{3x}\cos(x) - \left(-3e^{3x}(\sin(x)) + 9\int e^{3x}\sin(x)dx\right)$$

$$= -e^{3x}\cos(x) + 3e^{3x}\sin(x) - 9\int e^{3x}\sin(x)dx$$

You can see that both the left- and right-hand sides of this equation contains the original question, let's call it *I*. So, after re-expressing the result in terms of *I* you get:

$$I = -e^{3x}\cos(x) + 3e^{3x}\sin(x) - 9I$$

So, after rearranging: $I = \int e^{3x}\sin(x)dx = \frac{-e^{3x}\cos(x) + 3e^{3x}\sin(x)}{10} + c$

o. $\int x^2 \ln(4x)dx$ is suitable for integration by parts with $u = \ln(4x)$ and $\frac{dv}{dx} = x^2$.

Answer Space

What is $\int x^2 \ln(4x)dx$

Use integration by parts

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

$$\int x^2 \ln(4x)dx = \ln(4x) \cdot \frac{x^3}{3} - \int \left(\frac{x^3}{3}\right) \cdot \frac{1}{x} dx$$

$$\int x^2 \ln(4x)dx = \ln(4x) \cdot \frac{x^3}{3} - \left(\frac{x^3}{9}\right)$$

$$\int x^2 \ln(4x)dx = \ln(4x) \cdot \frac{x^3}{3} - \frac{x^3}{9} + c$$

Exploring Space

Let $u = \ln(4x)$, so $\frac{dv}{dx} = x^2$

For the formula you need u, du/dx and v:

$\frac{du}{dx} = \frac{1}{x}$ and $v = \int x^2 dx = \frac{x^3}{3} + c$

Arrows can help with the substitution.

The indefinite integral on the right hand side is:

$$\int \left(\frac{x^3}{3}\right) \cdot \frac{1}{x} dx = \int \frac{x^2}{3} dx = \int \frac{1}{3} x^2 dx = \frac{x^3}{9} + c$$

Using brackets can help with the minus signs.

Open the brackets to find the answer.

Check you are correct by differentiating your answer using the product rule, this should give your original integrand.