Learning Enhancement Team

University of East Anglia
STUDENT SUPPORT SERVICE

Steps into Calculus

Integration and Natural Logarithms

This guide describes an extremely useful substitution to help you integrate certain functions to give a natural logarithmic function. It describes a pattern you should learn to recognise and how to use it effectively.

Introduction

One of the main differences between differentiation and integration is that, in differentiation the rules are clear-cut. In differentiation if you know how a complicated function is made then you can chose an appropriate rule to differentiate it (see study guides: *More Complicated Functions*, *Differentiating Using the Power Rule*, *Differentiating Basic Functions*, *The Chain Rule*, *The Product Rule* and *The Quotient Rule*). This is not true for integration. Although it is important that you learn how to integrate basic functions (see study guides: *Integrating Using the Power Rule* and *Integrating Basic Functions*) beyond this, there are many different methods for integration which you should learn. Integrating more complicated functions becomes a matter of looking carefully at the integral you need to perform and trying a relevant method.

Many of these methods involve recognizing a pattern within an integral. One very common pattern you should be aware of is the following:

$$\int \frac{f'(x)}{f(x)}dx = \ln(f(x)) + c$$

where $f'(x)$ is used to denote the derivative of the function $f(x)$. Here the **integrand** (i.e. the piece of mathematics you are integrating) is a **fraction where the numerator is precisely the derivative of the denominator**. Because of this **you need an understanding of differentiation to use this rule correctly**. If your integral takes this form then the answer is the natural logarithm of the denominator. This integral plays an important role in science and it appears, for example, in exponential decay and growth and first order rate kinetics.

A special case of this integral is well-known. When $f(x) = x$, $f'(x) = 1$ and so:

$$\int \frac{1}{x}dx = \ln(x) + c$$

this integral fits the pattern with $f(x) = x$ and $f'(x) = 1$.

Example: What is the integral of $\frac{5}{5x-2}$ with respect to *x*?

The question can be written mathematically as $\int \frac{5}{5x-2}dx$.

This integral looks difficult but it fits the pattern on the first page of this guide as the derivative of the denominator $f(x)=5x-2$ is equal to the numerator 5. So:

$$\int \frac{5}{5x-2}dx = \ln(5x-2)+c$$

You can check this result by using the chain rule to differentiate the answer.

Example: What is the integral of $\frac{2x+3}{x^2+3x+3}$ with respect to *x*?

The question can be written mathematically as $\int \frac{2x+3}{x^2+3x+3}dx$.

This integral looks complicated until you recognise that it fits the pattern on the first page of this guide as the derivative of the denominator $f(x)=x^2+3x+3$ is $2x+3$. So:

$$\int \frac{2x+3}{x^2+3x+3}dx = \ln\left(x^2+3x+3\right)+c$$

Again you can check this result by using the chain rule to differentiate the answer.

Example: What is the integral of $-\tan x$ with respect to *x*?

The question can be written mathematically as $\int -\tan x dx$.

Often the pattern on the first page of this guide is hidden by other mathematics. Here you can use the trigonometric identity $\tan x = \frac{\sin x}{\cos x}$ to show that:

$$\int -\tan x dx = \int \frac{-\sin x}{\cos x}dx$$

Applying this trigonometric identity means you can re-express the integral with the derivative of the denominator $f(x)=\cos x$ being the numerator, $-\sin x$. So:

$$\int -\tan x dx = \int \frac{-\sin x}{\cos x}dx = \ln(\cos x)+c$$

Again you can check this result by using the chain rule to differentiate the answer.

Using a constant to adjust the integral

In order to use the pattern on the first page of this guide, **the numerator must strictly be the derivative of the denominator**. However you **can** adjust the numerator of the integrand to help it fit the pattern. **The only way you can adjust the numerator is by multiplication or division by a constant.** You cannot add or subtract a constant to the numerator. Also you cannot manipulate the numerator by using a variable i.e. you cannot add or subtract a variable and you cannot multiply or divide by a variable, for example *x*.

In summary:

- **Allowed** — Multiplying the numerator by a constant
Dividing the numerator by a constant
- **Not Allowed** — Adding or subtracting a constant or variable to the numerator
Multiplying or dividing the numerator by a variable

If you need to adjust the numerator of the integrand then you must ensure that you do not affect the overall integral. For example if you **multiply the numerator of the integrand by a constant** then you must **divide the integral by the same constant**. The multiplication and division cancel each other and you essentially do nothing to the integral. Similarly if you **divide the numerator of the integrand by a constant** then you must **multiply the integral by the same constant**. This is best illustrated by an example.

Example: What is the integral of $\frac{x}{x^2+7}$ with respect to *x*?

The question can be written mathematically as $\int \frac{x}{x^2+7}\,dx$.

The derivative of the denominator x^2+7 is $2x$ but you have a numerator of *x*. If you multiply the integrand by 2 then the integral would fit the pattern on page one:

$$\int \frac{x}{x^2+7}\,dx \xrightarrow{\times 2} \int \frac{2x}{x^2+7}\,dx$$

However you cannot make this adjustment on its own as you will change the integral by a factor of 2, i.e. you have altered the integral you are performing. However if you also divide the integral by two then you have made no overall change to your integral:

$$\int \frac{x}{x^2+7}\,dx \xrightarrow{\times 2 \text{ and } \div 2} \frac{1}{2}\int \frac{2x}{x^2+7}\,dx$$

The integrand now fits the pattern on page one and so:

$$\int \frac{x}{x^2+7}dx = \frac{1}{2}\int \frac{2x}{x^2+7}dx = \frac{1}{2}\ln\left(x^2+7\right)+c$$

You should note that the divide by 2 (the half) is carried forward into the answer. You can check this result by using the chain rule to differentiate the answer.

Example: What is the integral of $\frac{x^2-2}{6x-x^3}$ with respect to x?

The question can be written mathematically as $\int \frac{x^2-2}{6x-x^3}dx$.

The derivative of the denominator is $6-3x^2$ which is the numerator multiplied by -3 (it is very common to change a sign in the numerator by multiplying by -1). So:

$$\int \frac{x^2-2}{6x-x^3}dx \xrightarrow{\times -3 \text{ and } \div -3} -\frac{1}{3}\int \frac{6-3x^2}{6x-x^3}dx$$

The integrand now fits the pattern on page one and so:

$$\int \frac{x^2-2}{6x-x^3}dx = -\frac{1}{3}\int \frac{6-3x^2}{6x-x^3}dx = -\frac{1}{3}\ln\left(6x-x^3\right)+c$$

you can check this result by using the chain rule to differentiate the answer.

Want to know more?

If you have any further questions about this topic you can make an appointment to see a **Learning Enhancement Tutor** in the **Student Support Service**, as well as speaking to your lecturer or adviser.

Call: 01603 592761
Ask: ask.let@uea.ac.uk
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