# Learning Enhancement Team

## *Model answers:* Integration and Natural Logarithms

Integration and Natural Logarithms study guide

The answer in this worksheet use the following pattern to solve the problems:

$$\int \frac{f'(x)}{f(x)}\,dx = \ln(f(x)) + c$$

1.

(a) $\int \frac{2}{2x+5}\,dx = \ln(2x+5) + c$

As the derivative of $f(x) = 2x+5$ is $f'(x) = 2$, the integral fits the pattern of $\int \frac{f'(x)}{f(x)}\,dx$ and so the answer is the natural logarithm of the denominator.

(b) $\int \frac{3-4x}{6+3x-2x^2}\,dx = \ln(6+3x-2x^2) + c$

As the derivative of $f(x) = 6+3x-2x^2$ is $f'(x) = 3-4x$, the integral fits the pattern of $\int \frac{f'(x)}{f(x)}\,dx$ and so the answer is the natural logarithm of the denominator.

(c) $\int \frac{1-e^{-x}}{x+e^{-x}}\,dx = \ln(x+e^{-x}) + c$

As the derivative of $f(x) = x+e^{-x}$ is $f'(x) = 1-e^{-x}$, the integral fits the pattern of $\int \frac{f'(x)}{f(x)}\,dx$ and so the answer is the natural logarithm of the denominator.

(d) $\int \frac{1}{t \ln t} dt = \ln(\ln t) + c$

Don't be confused by the integral being performed with respect to *t*. The procedure works exactly the same. As the derivative of $f(t) = \ln t$ is $f'(t) = \frac{1}{t}$, the integral fits the pattern of $\int \frac{f'(t)}{f(t)} dt$ and so the answer is the natural logarithm of the denominator.

2. To calculate the definite integrals it is really useful to have a good understanding of the laws of logarithms, see study guide: *Laws of Logarithms*.

(a) $\int_3^4 \frac{1}{x-2} dx = \ln(2)$

As the derivative of $f(x) = x - 2$ is $f'(x) = 1$, the integral fits the pattern of $\int \frac{f'(x)}{f(x)} dx$ and so:

$$\int_3^4 \frac{1}{x-2} dx = [\ln(x-2)]_3^4$$

You now need to perform the definite integral. The study guide: *Definite Integrals* may help you if you find the following piece of mathematics difficult.

$$\int_3^4 \frac{1}{x-2} dx = [\ln(x-2)]_3^4$$
$$= \ln(4-2) - \ln(3-2) = \ln(2) - \ln(1) = \ln(2)$$

The result has used the fact that $\ln(1) = 0$.

(b) $\int_0^1 \frac{-2x}{3-x^2} dx = \ln\left(\frac{2}{3}\right)$

As the derivative of $f(x) = 3 - x^2$ is $f'(x) = -2x$, the integral fits the pattern of $\int \frac{f'(x)}{f(x)} dx$ and so:

$$\int_0^1 \frac{-2x}{3-x^2} dx = [\ln(3-x^2)]_0^1$$

You now need to perform the definite integral. The study guide: *Definite Integrals* may help you if you find the following piece of mathematics difficult.

$$\int_0^1 \frac{-2x}{3-x^2}\,dx = \left[\ln\left(3-x^2\right)\right]_0^1$$
$$= \ln\left(3-1^2\right) - \ln\left(3-0^2\right) = \ln(2) - \ln(3) = \ln\left(\frac{2}{3}\right)$$

The result has used the fact that $\ln A - \ln B = \ln\left(\frac{A}{B}\right)$.

(c) $\displaystyle\int_3^4 \frac{2x-4}{(x-2)^2}\,dx = \ln 4$

First open the bracket in the denominator to give $(x-2)^2 = x^2 - 4x + 4$ and so $f(x) = x^2 - 4x + 4$. As the derivative of $f(x)$ is $f'(x) = 2x - 4$, the integral fits the pattern of $\displaystyle\int \frac{f'(x)}{f(x)}\,dx$ and so:

$$\int_3^4 \frac{2x-4}{(x-2)^2}\,dx = \left[\ln(x-2)^2\right]_3^4 = \left[2\ln(x-2)\right]_3^4$$

as $\ln A^B = B\ln A$. You now need to perform the definite integral. The study guide: *Definite Integrals* may help you if you find the following piece of mathematics difficult.

$$\int_3^4 \frac{2x-4}{(x-2)^2}\,dx = \left[\ln(x-2)^2\right]_3^4 = \left[2\ln(x-2)\right]_3^4$$
$$= 2\ln(4-2) - 2\ln(3-2) = 2\ln(2) - 2\ln(1) = 2\ln(2) = \ln(4)$$

The result has used the fact that $\ln(1) = 0$ and $\ln A^B = B\ln A$.

(d) $\displaystyle\int_1^{1.5} \frac{3-2x}{3x\left(1-\frac{1}{3}x\right)}\,dx = \ln\left(\frac{9}{8}\right)$

First open the bracket in the denominator to give $3x\left(1-\frac{1}{3}x\right) = 3x - x^2$ and so $f(x) = 3x - x^2$. As the derivative of $f(x)$ is $f'(x) = 3 - 2x$, the integral fits the pattern of $\displaystyle\int \frac{f'(x)}{f(x)}\,dx$ and so:

$$\int_1^{1.5} \frac{3-2x}{3x\left(1-\frac{1}{3}x\right)}dx = \left[\ln\left(3x-x^2\right)\right]_1^{1.5}$$

You now need to perform the definite integral. The study guide: *Definite Integrals* may help you if you find the following piece of mathematics difficult.

$$\int_1^{1.5} \frac{3-2x}{3x\left(1-\frac{1}{3}x\right)}dx = \left[\ln\left(3x-x^2\right)\right]_1^{1.5}$$

$$= \ln\left(3\cdot(1.5)-\left(1.5^2\right)\right)-\ln\left(3\cdot 1-1^2\right) = \ln\left(\frac{9}{4}\right)-\ln 2 = \ln\left(\frac{9}{8}\right)$$

The result has used the fact that $\ln A - \ln B = \ln\left(\frac{A}{B}\right)$.

3.

(a) $\int \frac{x^2}{1-x^3}dx = -\frac{1}{3}\ln\left(1-x^3\right)+c$

As the derivative of $f(x)=1-x^3$ is $f'(x)=-3x^2$, you need to multiply the numerator by $-3$ in order to make the integral fit the pattern $\int \frac{f'(x)}{f(x)}dx$. You also need to divide the integral by $-3$ to balance the multiplication and so:

$$\int \frac{x^2}{1-x^3}dx = -\frac{1}{3}\int \frac{-3x^2}{1-x^3}dx$$

Now the integral fits the pattern and the answer is the natural logarithm of the denominator multiplied by $-1/3$.

(b) $\int \tan(2\theta)d\theta = -\frac{1}{2}\ln(\cos(2\theta))+c$

To solve this you first note that $\tan(2\theta)=\frac{\sin(2\theta)}{\cos(2\theta)}$ and also that the integral is with respect to $\theta$. So:

$$\int \tan(2\theta)d\theta = \int \frac{\sin(2\theta)}{\cos(2\theta)}d\theta$$

As the derivative of $f(\theta)=\cos(2\theta)$ is $f'(x)=-2\sin(2\theta)$, you need to multiply the numerator by $-2$ in order to make the integral fit the pattern $\int \frac{f'(x)}{f(x)}dx$. You also need to divide the integral by $-2$ to balance the multiplication and so:

$$\int \tan(2\theta)d\theta = -\frac{1}{2}\int \frac{-2\sin(2\theta)}{\cos(2\theta)}d\theta$$

Now the integral fits the pattern and the answer is the natural logarithm of the denominator multiplied by $-1/2$.

(c) $\int \frac{15x^3}{3x^4+2}dx = \frac{5}{4}\ln(3x^4+2)+c$

As the derivative of $f(x)=3x^4+2$ is $f'(x)=12x^3$, you need to multiply the numerator by $4/5$ in order to make the integral fit the pattern $\int \frac{f'(x)}{f(x)}dx$. You also need to divide the integral by $4/5$ to balance the multiplication and so:

$$\int \frac{15x^3}{3x^4+2}dx = \frac{5}{4}\int \frac{12x^3}{3x^4+2}dx$$

Now the integral fits the pattern and the answer is the natural logarithm of the denominator multiplied by $5/4$.

(d) $\int \frac{3e^{2t}+3}{e^{2t}+2t}dt = \frac{2}{3}\ln(e^{2t}+2t)+c$

Note that this integral is with respect to *t*. As the derivative of $f(t)=e^{2t}+2t$ is $f'(t)=2e^{2t}+2$ you need to multiply the numerator by $3/2$ in order to make the integral fit the pattern $\int \frac{f'(x)}{f(x)}dx$. You also need to divide the integral by $3/2$ to balance the multiplication and so:

$$\int \frac{3e^{2t}+3}{e^{2t}+2t}dt = \frac{2}{3}\int \frac{2e^{2t}+2}{e^{2t}+2t}dt$$

Now the integral fits the pattern and the answer is the natural logarithm of the denominator multiplied by $2/3$.

4. Calculate the definite integrals:

(a) $\int_{-2}^{-1} \frac{1}{3-x}\,dx = \ln\left(\frac{5}{4}\right)$

As the derivative of $f(x) = 3 - x$ is $f'(x) = -1$, you need to multiply the numerator by $-1$ in order to make the integral fit the pattern $\int \frac{f'(x)}{f(x)}\,dx$. You also need to divide the integral by $-1$ (which is identical to multiplying by $-1$) to balance the multiplication and so:

$$\int_{-2}^{-1} \frac{1}{3-x}\,dx = -\int_{-2}^{-1} \frac{-1}{3-x}\,dx = [-\ln(3-x)]_{-2}^{-1}$$

Now, putting in the limits:

$$\int_{-2}^{-1} \frac{1}{3-x}\,dx = -\int_{-2}^{-1} \frac{-1}{3-x}\,dx = [-\ln(3-x)]_{-2}^{-1}$$
$$= [-\ln(3-(-1))] - [-\ln(3-(-2))] = -\ln 4 + \ln 5 = \ln\left(\frac{5}{4}\right)$$

The result has used the fact that $\ln A - \ln B = \ln\left(\frac{A}{B}\right)$.

(b) $\int_{5}^{6} \frac{2}{x-3}\,dx = 2\ln\left(\frac{3}{2}\right)$

As the derivative of $f(x) = x - 3$ is $f'(x) = 1$, you need to multiply the numerator by $1/2$ in order to make the integral fit the pattern $\int \frac{f'(x)}{f(x)}\,dx$. You also need to divide the integral by $1/2$ to balance the multiplication and so:

$$\int_{5}^{6} \frac{2}{x-3}\,dx = 2\int_{5}^{6} \frac{1}{x-3}\,dx$$

Now, performing the integral and putting in the limits:

$$\int_5^6 \frac{2}{x-3}dx = 2\int_5^6 \frac{1}{x-3}dx = [2\ln(x-3)]_5^6$$
$$= 2\ln(6-3) - 2\ln(5-3) = 2\ln(3) - 2\ln(2)$$
$$= 2[\ln(3) - \ln(2)]$$
$$= 2\ln\left(\frac{3}{2}\right)$$

The result has used the fact that $\ln A - \ln B = \ln\left(\frac{A}{B}\right)$.

(c) $\int_0^2 \frac{x^2+1}{x^3+3x+7}dx = \frac{1}{3}\ln 3$

As the derivative of $f(x) = x^3 + 3x + 7$ is $f'(x) = 3x^2 + 3$, you need to multiply the numerator by 3 in order to make the integral fit the pattern $\int \frac{f'(x)}{f(x)}dx$. You also need to divide the integral by 3 to balance the multiplication and so:

$$\int_0^2 \frac{x^2+1}{x^3+3x+7}dx = \frac{1}{3}\int_0^2 \frac{3x^2+3}{x^3+3x+7}dx$$

Now, performing the integral and putting in the limits:

$$\int_0^2 \frac{x^2+1}{x^3+3x+7}dx = \frac{1}{3}\int_0^2 \frac{3x^2+3}{x^3+3x+7}dx = \left[\frac{1}{3}\ln(x^3+3x+7)\right]_0^2$$
$$= \frac{1}{3}\ln(2^3+3\cdot 2+7) - \frac{1}{3}\ln(0^3+3\cdot 0+7)$$
$$= \frac{1}{3}\ln(21) - \frac{1}{3}\ln(7) = \frac{1}{3}[\ln(21) - \ln(7)]$$
$$= \frac{1}{3}\ln(3)$$

The result has used the fact that $\ln A - \ln B = \ln\left(\frac{A}{B}\right)$.

(d) $\int_0^{\pi/2} \tan\left(\frac{\theta}{3}\right)d\theta = -3\ln\left(\sqrt{3}/2\right)$

In a similar manner to 3b, to solve this you must first note that $\tan(\theta/3) = \frac{\sin(\theta/3)}{\cos(\theta/3)}$ and also that the integral is with respect to $\theta$. So:

$$\int_0^{\pi/2} \tan(\theta/3)d\theta = \int_0^{\pi/2} \frac{\sin(\theta/3)}{\cos(\theta/3)}d\theta$$

As the derivative of $f(\theta) = \cos(\theta/3)$ is $f'(x) = -\frac{1}{3}\sin(\theta/3)$, you need to multiply the numerator by $-1/3$ in order to make the integral fit the pattern $\int \frac{f'(x)}{f(x)}dx$. You also need to divide the integral by $-1/3$ to balance the multiplication and so:

$$\int_0^{\pi/2} \tan(\theta/3)d\theta = \int_0^{\pi/2} \frac{\sin(\theta/3)}{\cos(\theta/3)}d\theta = -3\int_0^{\pi/2} \frac{-\sin(\theta/3)}{3\cos(\theta/3)}d\theta$$

You can now perform the integral and get the result $\ln(f(\theta))$:

$$\int_0^{\pi/2} \tan(\theta/3)d\theta = [-3\ln(\cos(\theta/3))]_0^{\pi/2}$$

So, inputting the limits gives (remember that these calculations are performed in **radians**):

$$\int_0^{\pi/2} \tan(\theta/3)d\theta = [-3\ln(\cos(\theta/3))]_0^{\pi/2} = -3\ln(\cos(\pi/6)) - (-3\ln(\cos 0))$$

As $\cos(0) = 1$, $\ln(\cos(0)) = \ln(1) = 0$ and $\cos(\pi/6) = \sqrt{3}/2$

$$\int_0^{\pi/2} \tan(\theta/3)d\theta = -3\ln(\sqrt{3}/2) + 0 = -3\ln(\sqrt{3}/2)$$

5.

(a) $\int \frac{x^2+2}{2}dx$

You can break the integrand up into two fractions as $\frac{x^2+2}{2} = \frac{x^2}{2} + 1$, each of which can be integrated using rules for basic functions, see study guide: *Integrating Basic Functions*. For completeness:

$$\int \frac{x^2+2}{2}dx = \int \frac{x^2}{2}dx + \int 1dx = \frac{x^3}{6} + x + c$$

(b) $\int \frac{3-x^2}{-2x}\,dx$

You can break the integrand up into two fractions as $\frac{3-x^2}{-2x} = -\frac{3}{2x} + \frac{x}{2}$, each of which can be integrated using rules for basic functions, see study guide: *Integrating Basic Functions*.

$$\int \frac{3-x^2}{-2x}\,dx = -\int \frac{3}{2x}\,dx + \int \frac{x}{2}\,dx = -\frac{3}{2}\ln(x) + \frac{x^2}{4} + c$$

(c) $\int \frac{x+3}{x^2-9}\,dx = \ln(x-3) + c$

You should notice that the denominator of this function is the difference of two squares which factorises to give $x^2 - 9 = (x+3)(x-3)$ (see study guide: *Factorising Quadratic Expressions*) so:

$$\int \frac{x+3}{x^2-9}\,dx = \int \frac{x+3}{(x+3)(x-3)}\,dx$$

which shows that the $x+3$ cancels down to give:

$$\int \frac{1}{x-3}\,dx$$

As the derivative of $f(x) = x-3$ is $f'(x) = 1$, the integral fits the pattern of $\int \frac{f'(x)}{f(x)}\,dx$ and so the answer is the natural logarithm of the denominator.

(d) $\int \frac{9x^2}{3x^2+x}\,dx$

You can factorise the denominator as there is a common factor of $x$ so $3x^2 + x = x(3x+1)$ this gives:

$$\int \frac{9x^2}{3x^2+x}\,dx = \int \frac{9x^2}{x(3x+1)}\,dx$$

This shows that the $x$ can cancel to give:

$$\int \frac{9x^2}{x(3x+1)}\,dx = \int \frac{9x}{3x+1}\,dx$$

Which is an integral which cannot be performed using the rule at the beginning of these model answers at the moment. However you can use a clever method to help perform this integral. Firstly add and subtract 3 from the numerator, this is allowed as adding and subtracting 3 is the same as adding zero, you get:

$$\int \frac{9x}{3x+1}dx = \int \frac{9x+3-3}{3x+1}dx$$

You can now split this integral into:

$$\int \frac{9x+3-3}{3x+1}dx = \int \frac{9x+3}{3x+1}dx - \int \frac{3}{3x+1}dx = \int \frac{3(3x+1)}{3x+1}dx - \int \frac{3}{3x+1}dx$$

In the first of these integrals you can cancel $3x+1$ to leave a simple integral of 3.

The second integral can be performs using the rule as the derivative of $f(x) = 3x+1$ is $f'(x) = 3$, the integral fits the pattern of $\int \frac{f'(x)}{f(x)}dx$ and so the answer to the second integral is the natural logarithm of the denominator. So:

$$\int \frac{3(3x+1)}{3x+1}dx - \int \frac{3}{3x+1}dx = \int 3dx - \int \frac{3}{3x+1}dx = 3x + \ln(3x+1) + c$$