Learning Enhancement Team

University of East Anglia
STUDENT SUPPORT SERVICE

Steps into Calculus

Using Trigonometric Formulas in Integration

This guide outlines some useful methods in integration which use trigonometric formulas. It shows how these formulas can be used to simplify some seemingly complicated integrals involving sines and cosines.

Introduction

Integration is a rich and varied subject which proves to be more intricate and wide ranging than its counterpart **differentiation**. By learning a few basic rules you can differentiate almost every function (see study guides: *Differentiating Basic Functions*, *The Chain Rule*, *The Product Rule*, *The Quotient Rule*). Integrating commonplace functions is not so straightforward and to become proficient in integration you need to have a wider range of techniques at your disposal (see study guides: *Integrating Basic Functions*, *Definite Integrals*, *Integration by Parts*, *Integration and Natural Logarithms*). The mathematical subject of trigonometry (the study of triangles, see: *Steps into Trigonometry*) offers you many useful relationships, in the form of **trigonometric formulas** or **identities**, which help solve seemingly difficult integrals. It is helpful to learn to recognise the form of integrals where the following methods apply and to have a good knowledge of a variety of helpful trigonometric formulas. The factsheet: *Trigonometric Identities* can help with this.

Basics integrals of sine and cosine

Many of the methods in this guide will produce an integral of either the sine or cosine function and it is worth reminding yourself of the following indefinite integrals:

$$\int a\sin kx\,dx = -\frac{a}{k}\cos kx + c$$

$$\int a\cos kx\,dx = \frac{a}{k}\sin kx + c$$

Double angle formulas

The **double angle formulas** are used to convert the integrals of either $\sin^n x$ or $\cos^n x$ **where *n* is a positive, even number**. When $n = 2$ you can use the formulas directly but when n is larger you will have to expand brackets and use the formulas repeatedly. The double angle formulas are:

$\sin^2 A = \frac{1}{2} - \frac{1}{2}\cos 2A$	use this with $\sin^n A$	(n is even)
$\cos^2 A = \frac{1}{2} + \frac{1}{2}\cos 2A$	use this with $\cos^n A$	(n is even)

Example: Find $\int \cos^2 x dx$.

Making a direct substitution using the double angle formula $\cos^2 A = \frac{1}{2} + \frac{1}{2}\cos 2A$ with $A = x$ you get:

$$\int \cos^2 x dx = \int \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right) dx$$

You can split this integral into two separate integrals:

$$\int \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right) dx = \int \tfrac{1}{2} dx + \int \tfrac{1}{2}\cos 2x dx$$

The first integral is found using the power rule and the second integral is the basic cosine integral on the first page of this guide with $a = 1$ and $k = 2$. Therefore:

$$\int \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right) dx = \int \tfrac{1}{2} dx + \int \tfrac{1}{2}\cos 2x dx = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + c$$

You can check your answer back by differentiation.

Example: Find $\int \cos^4 x dx$.

Firstly, you can rewrite $\cos^4 x$ as $\left(\cos^2 x\right)^2$. Now you can use the double angle formula on $\cos^2 x$ in a similar way to the previous question. So, after expanding the brackets you get:

$$\left(\cos^2 x\right)^2 = \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right)\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right) = \tfrac{1}{4} + \tfrac{1}{2}\cos 2x + \tfrac{1}{4}\cos^2 2x$$

You can see that the third term is also a cosine squared term and so you can apply the double angle formula for a second time (now with $A = 2x$) to get:

$$\tfrac{1}{4}\cos^2 2x = \tfrac{1}{4}\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 4x\right) = \tfrac{1}{8} + \tfrac{1}{8}\cos 4x$$

And so, after using the two results above and adding the fractions, the integral can be written as:

$$\int \cos^4 x dx = \int \left(\tfrac{3}{8} + \tfrac{1}{2}\cos 2x + \tfrac{1}{8}\cos 4x\right) dx$$

You can split this integral into three separate integrals:

$$\int \left(\tfrac{3}{8} + \tfrac{1}{2}\cos 2x + \tfrac{1}{8}\cos 4x\right) dx = \int \tfrac{3}{8} dx + \int \tfrac{1}{2}\cos 2x dx + \int \tfrac{1}{8}\cos 4x dx$$

The first integral is found using the power rule. The second and third integrals are basic cosine integrals (given on the first page of this guide) and so:

$$\int \cos^4 x dx = \int \tfrac{3}{8} dx + \int \tfrac{1}{2}\cos 2x dx + \int \tfrac{1}{8}\cos 4x dx$$

$$= \tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + c$$

You can check your answer back by differentiation.

Product formulas

You can use trigonometric formulas to help you integrate the result of multiplying basic sine and cosine functions together by using the **product formulas** shown below.

$$\sin A \sin B = \tfrac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$$
$$\cos A \cos B = \tfrac{1}{2}\left(\cos(A-B) + \cos(A+B)\right)$$
$$\sin A \cos B = \tfrac{1}{2}\left(\sin(A-B) + \sin(A+B)\right)$$

Usually A and B are multiples of x and so when they are added (or subtracted as necessary) you get a constant multiplied by x. You can use the product formulas to convert integrands which are products sines and/or cosines to additions or subtractions of sines or cosines. These in turn can be integrated using the basic integrals on the first page of this guide.

Example: Find $\int \sin 5x \sin 3x dx$

This is a product of two sine functions so use $\sin A \sin B = \tfrac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$ with $A = 5x$ and $B = 3x$ to help you with the integral. The integrand becomes:

$$\sin 5x \sin 3x = \tfrac{1}{2}\left(\cos(5x-3x) - \cos(5x+3x)\right) = \tfrac{1}{2}\cos 2x - \tfrac{1}{2}\cos 8x$$

and the two resulting functions can be integrated using the basic integral of cosine:

$$\int \sin 5x \sin 3x dx = \int \tfrac{1}{2}\cos 2x dx - \int \tfrac{1}{2}\cos 8x dx = \tfrac{1}{4}\sin 2x - \tfrac{1}{16}\sin 8x + c$$

You can check your answer back by differentiation.

Example: Find $\int \cos 3x \sin 5x dx$

This is the product of a sine function and a cosine function. You can use the formula $\sin A \cos B = \tfrac{1}{2}(\sin(A-B)+\sin(A+B))$ to help you with the integral. Looking carefully at this product formula you see that A is associated with the sine function and B is associated with the cosine function. So here $A = 5x$ and $B = 3x$ meaning that the integrand can be rewritten as:

$$\cos 3x \sin 5x = \tfrac{1}{2}(\sin(5x-3x)+\sin(5x+3x)) = \tfrac{1}{2}\sin 2x + \tfrac{1}{2}\sin 8x$$

These two new sine functions can be integrated in turn using the basic sine integral on the first page of this guide and so:

$$\int \cos 3x \sin 5x dx = \int \tfrac{1}{2}\sin 2x dx + \int \tfrac{1}{2}\sin 8x dx = -\tfrac{1}{4}\cos 2x - \tfrac{1}{16}\cos 8x + c$$

Again, using a trigonometric formula has transformed a seemingly difficult integral into two basic integrals. You can check your answer back by differentiation.

Want to know more?

If you have any further questions about this topic you can make an appointment to see a **Learning Enhancement Tutor** in the **Student Support Service**, as well as speaking to your lecturer or adviser.

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