Learning Enhancement Team

University of East Anglia

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Model answers: Using Trigonometric Formulas in Integration

These are the model answers for the worksheet that has questions on using trigonometric formulas in integration.

Using Trigonometric Formulas in Integration Study Guide

You can see that in both integrals $\int \sin^2(x)dx$ and $\int \sin^2(x)dx$ the power is a positive and even number. So to find these integrals you will need to use the double angle formulas.

a. $\int \sin^2(x)dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + c$

Making a direct substitution using the double angle formula $\sin^2 A = \frac{1}{2} - \frac{1}{2}\cos 2A$ with $A = x$ you get:

$$\int \sin^2(x)dx = \int \left(\tfrac{1}{2} - \tfrac{1}{2}\cos(2x)\right)dx$$

You can split this integral into two separate integrals:

$$\int \left(\tfrac{1}{2} - \tfrac{1}{2}\cos(2x)\right)dx = \int \tfrac{1}{2}dx - \int \tfrac{1}{2}\cos(2x)dx$$

The first integral is found using the power rule (to remind yourself of how to use the power rule you can check the study guide: *Integrating Using the Power Rule*) and the second integral is the basic cosine integral (to remind yourself of how to integrate basic trigonometric functions you can check the study guide: *Integrating Basic Functions*). Therefore:

$$\int \left(\tfrac{1}{2} - \tfrac{1}{2}\cos(2x)\right)dx = \int \tfrac{1}{2}dx - \int \tfrac{1}{2}\cos(2x)dx = \tfrac{1}{2}x - \tfrac{1}{4}\sin(2x) + c$$

You can check your answer back by differentiation.

b. $\int \sin^4(x)dx = \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + c$.

Firstly, you can rewrite $\sin^4(x)$ as $(\sin^2(x))^2$. Now you can use the double angle formula on $\sin^2(x)$ in a similar way to the previous question. So, after expanding the brackets you get:

$$(\sin^2(x))^2 = \left(\tfrac{1}{2} - \tfrac{1}{2}\cos(2x)\right)\left(\tfrac{1}{2} - \tfrac{1}{2}\cos(2x)\right) = \tfrac{1}{4} - \tfrac{1}{2}\cos(2x) + \tfrac{1}{4}\cos^2(2x)$$

You can see that the third term is a cosine squared term and so you can apply the double angle formula $\cos^2 A = \frac{1}{2} + \frac{1}{2}\cos 2A$ (now with $A = 2x$) to get:

$$\tfrac{1}{4}\cos^2(2x) = \tfrac{1}{4}\left(\tfrac{1}{2} + \tfrac{1}{2}\cos(4x)\right) = \tfrac{1}{8} + \tfrac{1}{8}\cos(4x)$$

And so, after using the two results above and adding the fractions, the integral can be written as:

$$\int \sin^4(x)dx = \int \left(\tfrac{3}{8} - \tfrac{1}{2}\cos(2x) + \tfrac{1}{8}\cos(4x)\right)dx$$

You can split this integral into three separate integrals:

$$\int \left(\tfrac{3}{8} - \tfrac{1}{2}\cos(2x) + \tfrac{1}{8}\cos(4x)\right)dx = \int \tfrac{3}{8}\,dx - \int \tfrac{1}{2}\cos(2x)dx + \int \tfrac{1}{8}\cos(4x)dx$$

The first integral is found using the power rule (to remind yourself of how to use the power rule you can check the study guide: *Integrating Using the Power Rule*). The second and third integrals are basic cosine integrals (to remind yourself of how to integrate basic trigonometric functions you can check the study guide: *Integrating Basic Functions*) and so:

$$\begin{aligned}\int \sin^4(x)dx &= \int \tfrac{3}{8}\,dx - \int \tfrac{1}{2}\cos(2x)dx + \int \tfrac{1}{8}\cos(4x)dx \\ &= \tfrac{3}{8}x - \tfrac{1}{4}\sin(2x) + \tfrac{1}{32}\sin(4x) + c\end{aligned}$$

You can check your answer back by differentiation.

2.

a. $\int \sin(4x)\sin(2x)dx = \frac{1}{4}\sin(2x) - \frac{1}{12}\sin(6x) + c$

This is a product of two sine functions so you use the following product formula:

$$\sin A \sin B = \tfrac{1}{2}\left(\cos(A-B) - \cos(A+B)\right) \text{ with } A = 4x \text{ and } B = 2x$$

to help you with the integral. The integrand becomes:

$$\sin(4x)\sin(2x) = \tfrac{1}{2}(\cos(4x-2x) - \cos(4x+2x)) = \tfrac{1}{2}\cos(2x) - \tfrac{1}{2}\cos(6x)$$

and the two resulting functions can be integrated using the basic integral of cosine (to remind yourself of how to integrate basic trigonometric functions you can check the study guide: *Integrating Basic Functions*):

$$\int \sin(4x)\sin(2x)dx = \int \tfrac{1}{2}\cos(2x)dx - \int \tfrac{1}{2}\cos(6x)dx = \tfrac{1}{4}\sin(2x) - \tfrac{1}{12}\sin(6x) + c$$

You can check your answer back by differentiation.

b. $\int \sin(12x)\sin(5x)dx = \tfrac{1}{14}\sin(7x) - \tfrac{1}{34}\sin(17x) + c$

This is a product of two sine functions so you use the following product formula:

$$\sin A \sin B = \tfrac{1}{2}(\cos(A-B) - \cos(A+B)) \text{ with } A = 12x \text{ and } B = 5x$$

to help you with the integral. The integrand becomes:

$$\sin(12x)\sin(5x) = \tfrac{1}{2}(\cos(12x-5x) - \cos(12x+5x)) = \tfrac{1}{2}\cos(7x) - \tfrac{1}{2}\cos(17x)$$

and the two resulting functions can be integrated using the basic integral of cosine (to remind yourself of how to integrate basic trigonometric functions you can check the study guide: *Integrating Basic Functions*):

$$\int \sin(12x)\sin(5x)dx = \int \tfrac{1}{2}\cos(7x)dx - \int \tfrac{1}{2}\cos(17x)dx = \tfrac{1}{14}\sin(7x) - \tfrac{1}{34}\sin(17x) + c$$

You can check your answer back by differentiation.

c. $\int \cos(12x)\cos(5x)dx = \tfrac{1}{14}\sin(7x) + \tfrac{1}{34}\sin(17x) + c$

This is a product of two cosine functions so you use the following product formula:

$$\cos A \cos B = \tfrac{1}{2}(\cos(A-B) + \cos(A+B)) \text{ with } A = 12x \text{ and } B = 5x$$

to help you with the integral. The integrand becomes:

$$\cos(12x)\cos(5x) = \tfrac{1}{2}(\cos(12x-5x) + \cos(12x+5x)) = \tfrac{1}{2}\cos(7x) + \tfrac{1}{2}\cos(17x)$$

and the two resulting functions can be integrated using the basic integral of cosine (to remind yourself of how to integrate basic trigonometric functions you can check the study guide: *Integrating Basic Functions*):

$$\int \cos(12x)\cos(5x)dx = \int \tfrac{1}{2}\cos(7x)dx + \int \tfrac{1}{2}\cos(17x)dx$$
$$= \tfrac{1}{14}\sin(7x) + \tfrac{1}{34}\sin(17x) + c$$

You can check your answer back by differentiation.

d. $\int \sin(x)\cos(x)dx$

This is the product of a sine and a cosine function so you use the following product formula:

$$\sin A\cos B = \tfrac{1}{2}(\sin(A-B)+\sin(A+B)) \text{ with } A = x \text{ and } B = x$$

to help you with the integral. The integrand becomes:

$$\sin(x)\cos(x) = \tfrac{1}{2}(\sin(x-x)+\sin(x+x)) = \tfrac{1}{2}\sin(0)+\tfrac{1}{2}\sin(2x) = \tfrac{1}{2}\sin(2x)$$

and the resulting function can be integrated using the basic integral of sine (to remind yourself of how to integrate basic trigonometric functions you can check the study guide: *Integrating Basic Functions*):

$$\int \sin(x)\cos(x)dx = \int \tfrac{1}{2}\sin(2x)dx = -\tfrac{1}{4}\cos(2x)+c$$

You can check your answer back by differentiation.

e. $\int \sin(\tfrac{3}{2}x)\sin(x)dx = \sin(\tfrac{1}{2}x) - \tfrac{1}{5}\sin(\tfrac{5}{2}x)+c$

This is a product of two sine functions so you use the following product formula:

$$\sin A\sin B = \tfrac{1}{2}(\cos(A-B)-\cos(A+B)) \text{ with } A = \tfrac{3}{2}x \text{ and } B = x$$

to help you with the integral. The integrand becomes:

$$\sin(\tfrac{3}{2}x)\sin(x) = \tfrac{1}{2}(\cos(\tfrac{3}{2}x-x)-\cos(\tfrac{3}{2}x+x)) = \tfrac{1}{2}\cos(\tfrac{1}{2}x)-\tfrac{1}{2}\cos(\tfrac{5}{2}x)$$

and the two resulting functions can be integrated using the basic integral of cosine (to remind yourself of how to integrate basic trigonometric functions you can check the study guide: *Integrating Basic Functions*):

$$\int \sin(\tfrac{3}{2}x)\sin(x)dx = \int \tfrac{1}{2}\cos(\tfrac{1}{2}x)dx - \int \tfrac{1}{2}\cos(\tfrac{5}{2}x)dx = \sin(\tfrac{1}{2}x)-\tfrac{1}{5}\sin(\tfrac{5}{2}x)+c$$

You can check your answer back by differentiation.

f. $\int \cos\left(\tfrac{4}{3}x\right)\cos\left(\tfrac{1}{6}x\right)dx = \tfrac{3}{7}\sin\left(\tfrac{7}{6}x\right)+\tfrac{1}{3}\sin\left(\tfrac{3}{2}x\right)+c$

This is a product of two cosine functions so you use the following product formula:

$$\cos A\cos B = \tfrac{1}{2}\left(\cos(A-B)+\cos(A+B)\right) \text{ with } A=\tfrac{4}{3}x \text{ and } B=\tfrac{1}{6}x$$

to help you with the integral. The integrand becomes:

$$\cos\left(\tfrac{4}{3}x\right)\cos\left(\tfrac{1}{6}x\right)=\tfrac{1}{2}\left(\cos\left(\tfrac{4}{3}x-\tfrac{1}{6}x\right)+\cos\left(\tfrac{4}{3}x+\tfrac{1}{6}x\right)\right)=\tfrac{1}{2}\cos\left(\tfrac{7}{6}x\right)+\tfrac{1}{2}\cos\left(\tfrac{3}{2}x\right)$$

and the two resulting functions can be integrated using the basic integral of cosine (to remind yourself of how to integrate basic trigonometric functions you can check the study guide: *Integrating Basic Functions*):

$$\int \cos\left(\tfrac{4}{3}x\right)\cos\left(\tfrac{1}{6}x\right)dx=\int \tfrac{1}{2}\cos\left(\tfrac{7}{6}x\right)dx+\int \tfrac{1}{2}\cos\left(\tfrac{3}{2}x\right)dx=\tfrac{3}{7}\sin\left(\tfrac{7}{6}x\right)+\tfrac{1}{3}\sin\left(\tfrac{3}{2}x\right)+c$$

You can check your answer back by differentiation.

g. $\int \sin\left(\tfrac{3}{4}x\right)\cos\left(\tfrac{1}{2}x\right)dx = -2\cos\left(\tfrac{1}{4}x\right)-\tfrac{2}{5}\cos\left(\tfrac{5}{4}x\right)+c$

This is the product of a sine and a cosine function so you use the following product formula:

$$\sin A\cos B=\tfrac{1}{2}\left(\sin(A-B)+\sin(A+B)\right) \text{ with } A=\tfrac{3}{4}x \text{ and } B=\tfrac{1}{2}x$$

to help you with the integral. The integrand becomes:

$$\sin\left(\tfrac{3}{4}x\right)\cos\left(\tfrac{1}{2}x\right)=\tfrac{1}{2}\left(\sin\left(\tfrac{3}{4}x-\tfrac{1}{2}x\right)+\sin\left(\tfrac{3}{4}x+\tfrac{1}{2}x\right)\right)=\tfrac{1}{2}\sin\left(\tfrac{1}{4}x\right)+\tfrac{1}{2}\sin\left(\tfrac{5}{4}x\right)$$

and the resulting functions can be integrated using the basic integral of sine (to remind yourself of how to integrate basic trigonometric functions you can check the study guide: *Integrating Basic Functions*):

$$\int \sin\left(\tfrac{3}{4}x\right)\cos\left(\tfrac{1}{2}x\right)dx=\int \tfrac{1}{2}\sin\left(\tfrac{1}{4}x\right)dx+\int \tfrac{1}{2}\sin\left(\tfrac{5}{4}x\right)dx=-2\cos\left(\tfrac{1}{4}x\right)-\tfrac{2}{5}\cos\left(\tfrac{5}{4}x\right)+c$$

You can check your answer back by differentiation.

h. $\int \cos\left(\tfrac{4}{3}x\right)\sin\left(\tfrac{11}{6}x\right)dx = -\cos\left(\tfrac{1}{2}x\right)-\tfrac{3}{19}\cos\left(\tfrac{19}{6}x\right)+c$

This is the product of a sine function and a cosine function. You can use the formula:

$$\sin A\cos B=\tfrac{1}{2}\left(\sin(A-B)+\sin(A+B)\right) \text{ to help you with the integral.}$$

Looking carefully at this product formula you see that A is associated with the sine function and B is associated with the cosine function. So here $A = \frac{11}{6}x$ and $B = \frac{4}{3}x$ meaning that the integrand can be rewritten as:

$$\cos\left(\tfrac{4}{3}x\right)\sin\left(\tfrac{11}{6}x\right) = \tfrac{1}{2}\left(\sin\left(\tfrac{11}{6}x - \tfrac{4}{3}x\right) + \sin\left(\tfrac{11}{6}x + \tfrac{4}{3}x\right)\right) = \tfrac{1}{2}\sin\left(\tfrac{1}{2}x\right) + \tfrac{1}{2}\sin\left(\tfrac{19}{6}x\right)$$

These two new sine functions can be integrated in turn using the basic sine integral (to remind yourself of how to integrate basic trigonometric functions you can check the study guide: *Integrating Basic Functions*):

$$\int \cos\left(\tfrac{4}{3}x\right)\sin\left(\tfrac{11}{6}x\right)dx = \int \tfrac{1}{2}\sin\left(\tfrac{1}{2}x\right)dx + \int \tfrac{1}{2}\sin\left(\tfrac{19}{6}x\right)dx = -\cos\left(\tfrac{1}{2}x\right) - \tfrac{3}{19}\cos\left(\tfrac{19}{6}x\right) + c$$

Again, using a trigonometric formula has transformed a seemingly difficult integral into two basic integrals. You can check your answer back by differentiation.

3. You can see that $\sin^2(x)\cos^2(x)$ can be rewritten as $(\sin(x)\cos(x))^2$ using the laws of indices (to remind yourself of the laws of indices you can check the study guide: *Laws of Indices*). Or, using double angle formulas you can rewrite them as $\left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)\left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right) = \left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right)\left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)$.

You need to find the integral $\int \sin^2(x)\cos^2(x)dx$. From the above you can see that you could find the integral either using the product formula or the double angle formulas.

First, using the product formula.

$$\sin^2(x)\cos^2(x) = (\sin(x)\cos(x))^2$$

And $\sin(x)\cos(x)$ is the product of a sine and a cosine function so you use the following product formula:

$$\sin A \cos B = \tfrac{1}{2}(\sin(A-B) + \sin(A+B)) \text{ with } A = x \text{ and } B = x$$

to help you with the integral. The integrand becomes:

$$\begin{aligned}\sin^2(x)\cos^2(x) &= (\sin(x)\cos(x))^2 \\ &= \left(\tfrac{1}{2}(\sin(x-x) + \sin(x+x))\right)^2 \\ &= \left(\tfrac{1}{2}\sin(0) + \tfrac{1}{2}\sin(2x)\right)^2 \\ &= \left(\tfrac{1}{2}\sin(2x)\right)^2 \\ &= \tfrac{1}{4}\sin^2(2x)\end{aligned}$$

Looking at the resulting function you can apply the double angle formula $\sin^2 A = \frac{1}{2} - \frac{1}{2}\cos 2A$ (now with $A = 2x$) to get:

$$\tfrac{1}{4}\sin^2(2x) = \tfrac{1}{4}\left(\tfrac{1}{2} - \tfrac{1}{2}\cos(4x)\right) = \tfrac{1}{8} - \tfrac{1}{8}\cos(4x)$$

and the resulting functions can be integrated using the power rule (to remind yourself of how to use the power rule you can check the study guide: *Integrating Using the Power Rule*) and the basic integral of cosine (to remind yourself of how to integrate basic trigonometric functions you can check the study guide: *Integrating Basic Functions*) and so:

$$\begin{aligned}\int \sin^2(x)\cos^2(x)dx &= \int (\sin(x)\cos(x))^2 dx \\ &= \int \tfrac{1}{4}\sin^2(2x)dx \\ &= \int \tfrac{1}{8}dx - \int \tfrac{1}{8}\cos(4x)dx \\ &= \tfrac{1}{8}x - \tfrac{1}{32}\cos(4x) + c\end{aligned}$$

You can check your answer back by differentiation.

Now you can solve the integral using double angle formulas you can rewrite $\sin^2(x)\cos^2(x)$ as $\left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right)\left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)$. So,

$$\begin{aligned}\sin^2(x)\cos^2(x) &= \left(\tfrac{1}{2} + \tfrac{1}{2}\cos(2x)\right)\left(\tfrac{1}{2} - \tfrac{1}{2}\cos(2x)\right) \\ &= \tfrac{1}{4} - \tfrac{1}{4}\cos(2x) + \tfrac{1}{4}\cos(2x) - \tfrac{1}{4}\cos^2(2x) \\ &= \tfrac{1}{4} - \tfrac{1}{4}\cos^2(2x)\end{aligned}$$

Looking at the resulting function you can apply the double angle formula $\cos^2 A = \frac{1}{2} + \frac{1}{2}\cos 2A$ (now with $A = 2x$) to get:

$$\tfrac{1}{4} - \tfrac{1}{4}\cos^2(2x) = \tfrac{1}{4} - \tfrac{1}{4}\left(\tfrac{1}{2} + \tfrac{1}{2}\cos(4x)\right) = \tfrac{1}{4} - \tfrac{1}{8} - \tfrac{1}{8}\cos(4x) = \tfrac{1}{8} - \tfrac{1}{8}\cos(4x)$$

and the resulting functions can be integrated using the power rule (to remind yourself of how to use the power rule you can check the study guide: *Integrating Using the Power Rule*) and the basic integral of cosine (to remind yourself of how to integrate basic trigonometric functions you can check the study guide: *Integrating Basic Functions*) and so:

$$\int \sin^2(x)\cos^2(x)dx = \int \left(\tfrac{1}{2} + \tfrac{1}{2}\cos(2x)\right)\left(\tfrac{1}{2} - \tfrac{1}{2}\cos(2x)\right)dx$$

$$= \int \left(\tfrac{1}{4} - \tfrac{1}{4}\cos^2(2x)\right)dx$$

$$= \int \left(\tfrac{1}{8} - \tfrac{1}{8}\cos(4x)\right)dx$$

$$= \int \tfrac{1}{8}dx - \int \tfrac{1}{8}\cos(4x)dx$$

$$= \tfrac{1}{8}x - \tfrac{1}{32}\cos(4x) + c$$

You can check your answer back by differentiation.

These model answers are one of a series on mathematics produced by the Learning Enhancement Team with funding from the UEA Alumni Fund. Scan the QR-code with a smartphone app for more resources.

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