

Definite Integrals

This guide shows you how to calculate definite integrals and how this can be used to find areas on graphs.

Introduction

The important mathematical subject of **calculus** comprises **differentiation** and **integration**. The study guide: [What is Integration?](#) describes two types of integration: **definite integration** and **indefinite integration**. These two types of integration are related but there are some important differences. Most importantly, the **result of a definite integral is a number** whereas **the result of an indefinite integral is a function** with a constant (usually given by c) added to it. You should read the study guide: [What is Integration?](#) to help you understand the terminology and notation used in integration.

A definite integral looks different to an indefinite integral because it is written with two numbers, one at the top and one at the bottom of the integral sign, which are called the **limits**. When evaluating definite integrals, you first perform the integration as you would do for indefinite integration, but then you input the limits into the answer. So, with definite integration, an extra step is required. Some basic techniques for finding indefinite integrals are given in the study guides: [Integrating Using the Power Rule](#), [Integrating Basic Functions](#), [Integration by Parts](#) and [Integration and Natural Logarithms](#). This study guide shows you how to substitute limits into definite integrals and how to evaluate the resulting mathematics. It also shows you that you can use the result of definite integration to find areas on graphs.

How to evaluate a definite integral

There are four steps needed in order to evaluate the definite integral of a given function $f(x)$ between the limits of $x = a$ and $x = b$:

Method to evaluate $\int_a^b f(x) dx$

1. Integrate the function using indefinite integration to give $f(x)$.
2. Put this answer and the limits in square brackets $[f(x)]_a^b$.
3. Evaluate the function at these limits to give $f(b)$ and $f(a)$.
4. Compute $f(b) - f(a)$ to give the value of the definite integral.

In order to understand how to perform definite integration, you should understand each step of this method.

1. First you perform the indefinite integration. The derivative of the function $f(x)$ can be written as $f'(x)$ so, as indefinite integration is the inverse of differentiation, the function $f'(x)$ integrates to $f(x) + c$. In other words:

$$\int f'(x) dx = f(x) + c$$

The “+ c” is not needed for definite integration and so you just need the $f(x)$ part.

2. The result $f(x)$ is put into square brackets [] and the limits are written at the top and bottom of the square brackets like this:

$$[f(x)]_a^b$$

You should notice that this new piece of notation has no integral sign, as the integration has already been performed, but the limits are still present as you have not used them yet.

3. To compute the value of the definite integral you substitute the upper limit into $f(x)$ to give $f(b)$ and the lower limit into $f(x)$ to give $f(a)$. You should read the study guide: [Using Functions](#) to help you with substituting limits into functions.
4. Subtract $f(a)$ from $f(b)$ to give $f(b) - f(a)$, this is the answer.

You can write this process mathematically as:

$$\int_a^b f'(x) dx = [f(x)]_a^b = f(b) - f(a)$$

Example: Evaluate the definite integral $\int_1^2 x^2 dx$.

This piece of mathematics can be interpreted as the definite integral of x^2 between the lower limit of $x = 1$ and the upper limit of $x = 2$.

1. The function you are integrating is $f'(x) = x^2$ and you can use the power rule for integration to perform it and so $f(x) = x^3 / 3$. You do not need the “+ c”.
2. Putting the result from step 1 and the upper and lower limits into $[f(x)]_a^b$ gives:

$$\int_1^2 x^2 dx = \left[\frac{x^3}{3} \right]_1^2$$

3. You now need to evaluate the function $f(x) = x^3 / 3$ at the lower limit of $x = 1$ and the upper limit of $x = 2$. Specifically:

$$\text{Upper Limit: } f(2) = \frac{2^3}{3} = \frac{8}{3}$$

$$\text{Lower Limit: } f(1) = \frac{1^3}{3} = \frac{1}{3}$$

4. The answer is given by $f(2) - f(1)$:

$$f(2) - f(1) = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

So the value of the definite integral is $7/3$. This can be written as:

$$\int_1^2 x^2 dx = \left[\frac{x^3}{3} \right]_1^2 = \left(\frac{2^3}{3} \right) - \left(\frac{1^3}{3} \right) = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

Example: Integrate the function $f'(x) = 12$ between the limits 0 and 3.

Here the function is $f'(x) = 12$, the lower limit is 0 and the upper limit is 3 and so you can write the integral as:

$$\int_0^3 12 dx$$

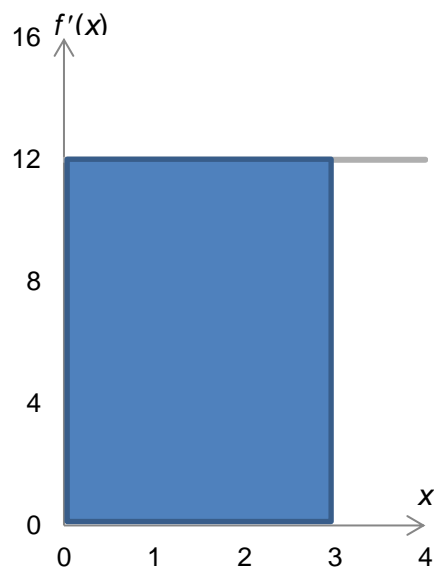
Step 1: $f'(x) = 12$ integrates to give $f(x) = 12x$

Step 2: $\int_0^3 12 dx = [12x]_0^3$

Step 3: $f(3) = 12 \times 3 = 36$ and $f(0) = 12 \times 0 = 0$

Step 4: $\int_0^3 12 dx = [12x]_0^3 = 12 \cdot 3 - 12 \cdot 0 = 36 - 0 = 36$

$$\text{and so } \int_0^3 12 dx = 36$$



The shaded area in the figure above shows the area under the graph of the function $f(x) = 12$ between $x = 0$, $x = 3$ and the x -axis. As the area of a rectangle is found by multiplying its width by its height, the area is $3 \times 12 = 36$. In this case the definite integral gives the area of a rectangle of height 12 and width 3.

Interpreting definite integrals as areas

It is true to say that all areas can be described by definite integrals but not all results of definite integrals are areas. It is important that you understand the difference between evaluating a definite integral and finding an area as the two are not always interchangeable.

1. If the function is entirely above the x-axis within the limits.

Definite integrals describe the region between the x-axis, the two limits and the function. **If the function is wholly above the x-axis between the limits then the result of the definite integral equals the area of the region it describes.** For instance, in the example above the shaded region has the same area as the result of the definite integral (36).

The definite integral of a function gives the area between the function, the x-axis and the limits *only if* the function is wholly positive i.e. wholly above the x-axis.

2. If the function is entirely below the x-axis within the limits.

When a function is wholly below the x-axis its value is negative and, as a result, the definite integral gives a negative result for this region. Consequently, as areas cannot be negative, the definite integral cannot describe an area. However **the positive value of the definite integral is equal to the area.** So all you need to do is to change the sign of the result (mathematically this is known as taking the **absolute value**).

If the function is wholly beneath the x-axis then the area between the limits, the x-axis and the function is the positive value of the definite integral.

Example: What is the area of the region enclosed by the x-axis, $f(x) = -3x^2$, and the limits $x = 1$ and $x = 2.5$?

Here the function is $f(x) = -3x^2$, the lower limit is 1 and the upper limit is 2.5 and so you can write the integral as:

$$\int_1^{2.5} -3x^2 dx$$

Step 1: $f(x) = -3x^2$ integrates using the power rule to give $f(x) = -x^3$

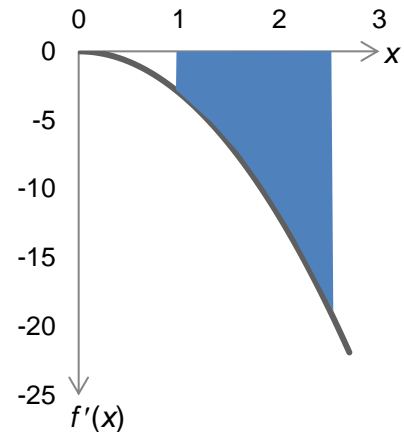
Step 2: $\int_1^{2.5} -3x^2 dx = \left[-x^3\right]_1^{2.5}$

Step 3: $f(2.5) = -2.5^3 = -15.625$ and $f(1) = -1^3 = -1$

Step 4: $\int_1^{2.5} -3x^2 dx = [-x^3]_1^{2.5} = -15.625 - (-1) = -14.625$

So $\int_1^{2.5} -3x^2 dx = -14.625$

The definite integral has a value of -14.625 which is negative. As the region is entirely beneath the x -axis then the area is the absolute value of this which is 14.625 .



3. If the function crosses the x -axis within the limits

If a function crosses the x -axis within the limits then a part of the function is positive and a part is negative. The portion above the x -axis contributes a positive value to the definite integral and the portion below the x -axis contributes a negative value to the definite integral. These will cancel each other out to some degree and so you cannot use the result of the definite integral to find the area. What you need to do is break down the integral into portions which are always above or below the axis and evaluate these integrals one-by-one. To find these portions you need to find the points where the function crosses the x -axis (these are called the **roots of the function**) and these form the new limits of the definite integral. For this you need to be able to sketch the graph of the function (see study guide: [Sketching a Graph](#)).

Example: Find the area between the function $f'(x) = x - x^3$, the x -axis, and the limits $x = -1$ and $x = 1$.

If you directly calculate the integral of the function between the limits -1 and 1 you get:

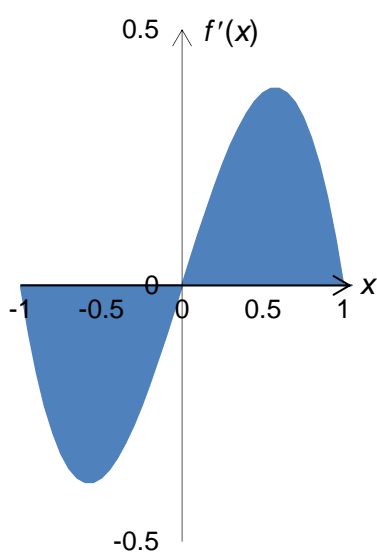
$$\int_{-1}^1 x - x^3 dx = \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_{-1}^1 = \left(\frac{1^2}{2} - \frac{1^4}{4} \right) - \left(\frac{(-1)^2}{2} - \frac{(-1)^4}{4} \right) = \left(\frac{1}{2} - \frac{1}{4} \right) - \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{1}{4} - \frac{1}{4} = 0$$

which seems to suggest that the area under the graph is 0. However the graph below shows that is not the case. To find the area you must calculate the definite integral for the parts of the graph above and below the x -axis separately. The part of the function above the x -axis is between $x = 0$ and $x = 1$. Using these as the limits gives:

$$\int_0^1 x - x^3 dx = \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \left(\frac{1^2}{2} - \frac{1^4}{4} \right) - \left(\frac{0^2}{2} - \frac{0^4}{4} \right) = \left(\frac{1}{2} - \frac{1}{4} \right) - (0) = \frac{1}{4}$$

The part of the graph that is below the x -axis is between $x = -1$ and $x = 0$ and so:

$$\int_{-1}^0 x - x^3 dx = \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_{-1}^0 = \left(\frac{0^2}{2} - \frac{0^4}{4} \right) - \left(\frac{(-1)^2}{2} - \frac{(-1)^4}{4} \right) = (0) - \left(\frac{1}{2} - \frac{1}{4} \right) = -\frac{1}{4}$$



These two results show that the definite integral of the part above the x -axis gave $1/4$ and the definite integral of the part below gave $-1/4$. So, when the definite integral was evaluated between $x = -1$ and $x = 1$, the two results cancel each other out.

To find the area you can treat the result of the definite integral for the part of the graph wholly above the x -axis, as equivalent to the area of that region, $1/4$. However you need to take the absolute value for the region wholly below the x -axis and so the definite integral result of $-1/4$ equates to an area of $+1/4$. Adding these two results gives a total area of $1/4 + 1/4 = 1/2$.

Want to know more?

If you have any further questions about this topic you can make an appointment to see a [Learning Enhancement Tutor](#) in the [Student Support Service](#), as well as speaking to your lecturer or adviser.

- 📞 Call: 01603 592761
- 💻 Ask: ask.let@uea.ac.uk
- 🔗 Click: <https://portal.uea.ac.uk/student-support-service/learning-enhancement>

There are many other resources to help you with your studies on our [website](#). For this topic, these include questions to [practise](#), [model solutions](#) and a [webcast](#).

Your comments or suggestions about our resources are very welcome.



Scan the QR-code with a smartphone app for a webcast of this study guide.

