Learning Enhancement Team

University of East Anglia
STUDENT SUPPORT SERVICE

Model Answers: Definite Integrals

These are the model answers for the worksheet that has question on definite integrals.

Definite Integrals study guide

1.

a. $\int_1^3 2\,dx$

This piece of mathematics can be interpreted as the definite integral of the function $f'(x) = 2$ between the lower limit of $x = 1$ and the upper limit of $x = 3$.

1. The function you are integrating is $f'(x) = 2$ and you can use the power rule for integration to perform it (you can check the study guide: *Integrating Using the Power Rule* to remind yourself of the power rule, and so $f(x) = 2x$. You do not need the "$+c$".

2. Putting the result from step 1 and the upper and lower limits into $[f(x)]_a^b$ gives:

$$\int_1^3 2\,dx = [2x]_1^3$$

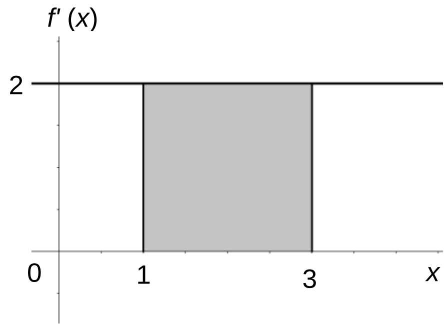
3. You now need to evaluate the function $f(x) = 2x$ at the lower limit of $x = 1$ and the upper limit of $x = 3$. Specifically:

Upper Limit: $f(3) = 2 \cdot 3 = 6$
Lower Limit: $f(1) = 2 \cdot 1 = 2$

4. The answer is given by $f(3) - f(1) = 6 - 2 = 4$. So the value of the definite integral is 4. This can be written as:

$$\int_1^3 2\,dx = [2x]_1^3 = (2 \cdot 3) - (2 \cdot 1) = 6 - 2 = 4$$

The shaded area in the figure shows the area under the graph of the function $f'(x) = 2$ between $x = 1$, $x = 3$ and the x-axis. As the area of a rectangle is found by multiplying its width by its height, the area is $2 \cdot 2 = 4$. In this case the definite integral gives the area of a rectangle of height 2 and width 2.



b. $\int_0^2 x\,dx$

This piece of mathematics can be interpreted as the definite integral of the function $f'(x) = x$ between the lower limit of $x = 0$ and the upper limit of $x = 2$.

1. The function you are integrating is $f'(x) = x$ and you can use the power rule for integration to perform it (you can check the study guide: *Integrating Using the Power Rule* to remind yourself of the power rule, and so $f(x) = x^2/2$. You do not need the
"$+c$".

2. Putting the result from step 1 and the upper and lower limits into $[f(x)]_a^b$ gives:

$$\int_0^2 x\,dx = \left[\frac{x^2}{2}\right]_1^2$$

3. You now need to evaluate the function $f(x) = x^2/2$ at the lower limit of $x = 0$ and the upper limit of $x = 2$. Specifically:

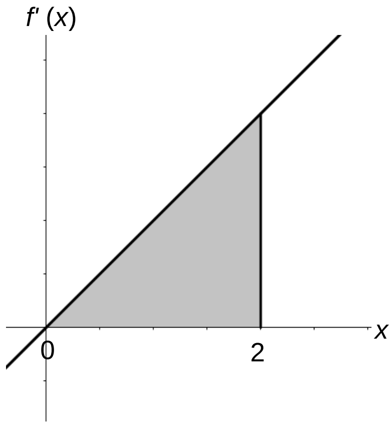
Upper Limit: $f(2) = \frac{2^2}{2} = 2$

Lower Limit: $f(0) = \frac{0^2}{2} = 0$

4. The answer is given by $f(2) - f(0) = 2 - 0 = 2$. So the value of the definite integral is 2. This can be written as:

$$\int_0^2 x\,dx = \left[\frac{x^2}{2}\right]_1^2 = 2 - 0 = 2$$

The shaded area in the figure shows the area under the graph of the function $f'(x)=x$ between $x=0$, $x=2$ and the *x*-axis.



As the area of a triangle is found by multiplying its width by its height and dividing it by 2, the area is $\frac{2 \cdot 2}{2}=2$. In this case the definite integral gives the area of a triangle of height 2 and width 2.

c. $\int_4^7 x^3\,dx$

This piece of mathematics can be interpreted as the definite integral of the function $f'(x)=x^3$ between the lower limit of $x=4$ and the upper limit of $x=7$.

1. The function you are integrating is $f'(x)=x^3$ and you can use the power rule for integration to perform it (you can check the study guide: *Integrating Using the Power Rule* to remind yourself of the power rule, and so $f(x)=x^4/4$. You do not need the "$+c$".

2. Putting the result from step 1 and the upper and lower limits into $[f(x)]_a^b$ gives:

$$\int_4^7 x^3\,dx=\left[\frac{x^4}{4}\right]_4^7$$

3. You now need to evaluate the function $f(x)=x^4/4$ at the lower limit of $x=4$ and the upper limit of $x=7$. Specifically:

Upper Limit: $f(7)=\frac{7^4}{4}=600.25$

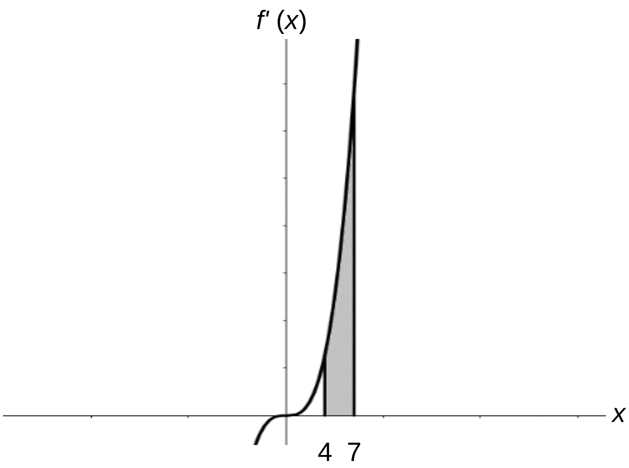
Lower Limit: $f(4)=\frac{4^4}{4}=64$

4. The answer is given by $f(7)-f(4)=600.25-64=536.25$

So the value of the definite integral is 536.25. This can be written as:

$$\int_4^7 x^3\,dx=\left[\frac{x^4}{4}\right]_4^7=600.25-64=536.25$$

The shaded area in the figure shows the area under the graph of the function $f'(x)=x^3$ between $x=4$, $x=7$ and the x-axis.



d. $\int_2^5 x^{-1}\,dx$

This piece of mathematics can be interpreted as the definite integral of the function $f'(x)=x^{-1}$ between the lower limit of $x=2$ and the upper limit of $x=5$.

1. The function you are integrating is $f'(x)=x^{-1}$ and you can use the power rule for integration to perform it (you can check the study guide: *Integrating Using the Power Rule* to remind yourself of the power rule, and so $f(x)=\ln x$. You do not need the "$+c$".

2. Putting the result from step 1 and the upper and lower limits into $[f(x)]_a^b$ gives:

$$\int_2^5 x^{-1}\,dx=[\ln x]_2^5$$

3. You now need to evaluate the function $f(x)=\ln x$ at the lower limit of $x=2$ and the upper limit of $x=5$. Specifically:

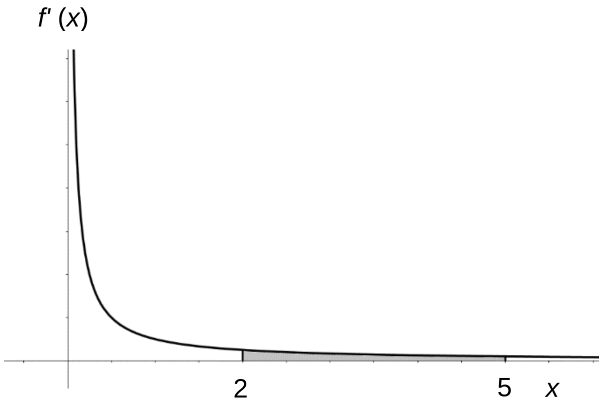
Upper Limit: $f(5)=\ln 5=1.61$ to 2 d.p.

Lower Limit: $f(2)=\ln 2=0.69$ to 2 d.p.

4. The answer is given by $f(5)-f(2)=\ln 5-\ln 2=0.92$. This can be written as:

$$\int_2^5 x^{-1}\,dx=[\ln x]_2^5=\ln 5-\ln 2=0.92 \text{ to 2 d.p.}$$

The shaded area in the figure shows the area under the graph of the function $f'(x) = x^{-1}$ between $x = 2$, $x = 5$ and the x-axis.



e. $\int_0^{\pi} \sin(x)dx$

This piece of mathematics can be interpreted as the definite integral of the function $f'(x) = \sin x$ between the lower limit of $x = 0$ and the upper limit of $x = \pi$.

1. The function you are integrating is $f'(x) = \sin x$ this is a trigonometric function. You can check the study guide: *Integrating Basic Functions* to remind yourself of rules for integrating trigonometric functions, and so $f(x) = -\cos x$. You do not need the "$+c$".

2. Putting the result from step 1 and the upper and lower limits into $[f(x)]_a^b$ gives:

$$\int_0^{\pi} \sin(x)dx = [-\cos x]_0^{\pi}$$

3. You now need to evaluate the function $f(x) = -\cos x$ at the lower limit of $x = 0$ and the upper limit of $x = \pi$. Specifically:

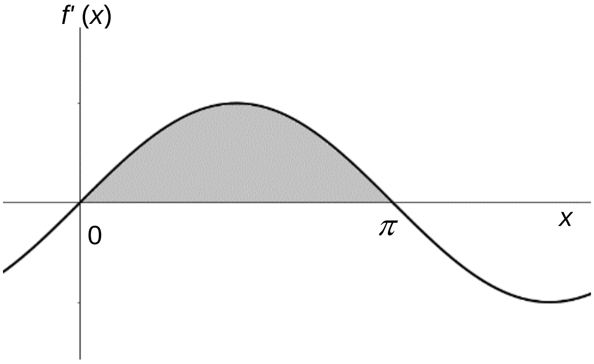
Upper Limit: $f(\pi) = -\cos \pi = -(-1) = 1$

Lower Limit: $f(0) = -\cos 0 = -(1) = -1$

4. The answer is given by $f(\pi) - f(0) = -1 - (-1) = 2$. This can be written as:

$$\int_0^{\pi} \sin(x)dx = [-\cos x]_0^{\pi} = -\cos \pi - (-\cos 0) = 2$$

The shaded area in the figure shows the area under the graph of the function $f'(x) = \sin(x)$ between $x = 0$, $x = \pi$ and the x-axis.



f. $\int_0^{\pi/2} \cos(x)dx$

This piece of mathematics can be interpreted as the definite integral of the function $f'(x) = \cos(x)$ between the lower limit of $x = 0$ and the upper limit of $x = \pi/2$.

1. The function you are integrating is $f'(x) = \cos(x)$ this is a trigonometric function. You can check the study guide: *Integrating Basic Functions* to remind yourself of rules for integrating trigonometric functions, and so $f(x) = \sin(x)$. You do not need the "$+c$".

2. Putting the result from step 1 and the upper and lower limits into $[f(x)]_a^b$ gives:

$$\int_0^{\pi/2} \cos(x)dx = [\sin x]_0^{\pi/2}$$

3. You now need to evaluate the function $f(x) = \sin(x)$ at the lower limit of $x = 0$ and the upper limit of $x = \pi/2$. Specifically:

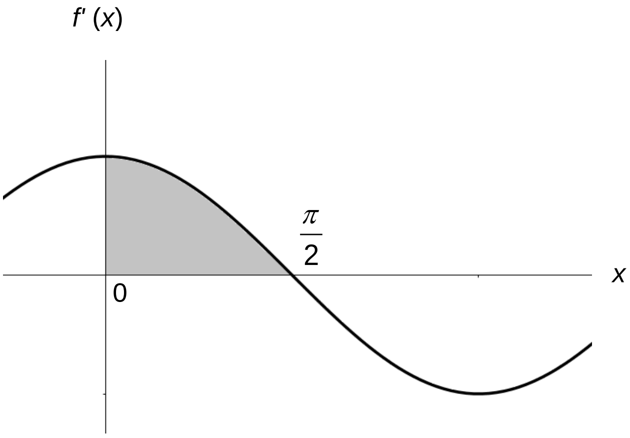
Upper Limit: $f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$

Lower Limit: $f(0) = \sin(0) = 0$

4. The answer is given by $f\left(\frac{\pi}{2}\right) - f(0) = 1 - 0 = 0$. This can be written as:

$$\int_0^{\pi/2} \cos(x)dx = [\sin x]_0^{\pi/2} = 1 - 0 = 1$$

The shaded area in the figure shows the area under the graph of the function $f'(x) = \cos(x)$ between $x = 0$, $x = \pi/2$ and the x-axis.



g. $\int_1^2 e^x \, dx$

This piece of mathematics can be interpreted as the definite integral of the function $f'(x) = e^x$ between the lower limit of $x = 1$ and the upper limit of $x = 2$.

1. The function you are integrating is $f'(x) = e^x$ this is an exponential function. You can check the study guide: *Integrating Basic Functions* to remind yourself of rules for integrating exponential functions, and so $f(x) = e^x$. You do not need the "$+c$".

2. Putting the result from step 1 and the upper and lower limits into $[f(x)]_a^b$ gives:

$$\int_1^2 e^x \, dx = \left[e^x\right]_1^2$$

3. You now need to evaluate the function $f(x) = e^x$ at the lower limit of $x = 1$ and the upper limit of $x = 2$. Specifically:

Upper Limit: $f(2) = e^2 = 7.39$ to 2 d.p.

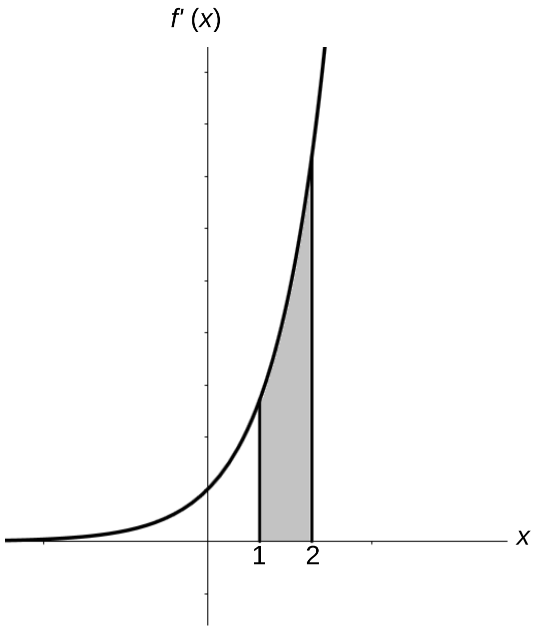
Lower Limit: $f(1) = e^1 = 2.72$ to 2 d.p.

4. The answer is given by $f(2) - f(1) = e^2 - e = 7.39 - 2.72 = 4.67$ to 2 d.p.

So the value of the definite integral is 4.67. This can be written as:

$$\int_1^2 e^x \, dx = \left[e^x\right]_1^2 = e^2 - e = 7.39 - 2.72 = 4.67 \text{ to 2 d.p.}$$

The shaded area in the figure shows the area under the graph of the function $f'(x) = e^x$ between $x = 1$, $x = 2$ and the x-axis.



h. $\int_1^4 \ln x \, dx$

This piece of mathematics can be interpreted as the definite integral of the function $f'(x) = \ln x$ between the lower limit of $x = 1$ and the upper limit of $x = 4$.

1. The function you are integrating is $f'(x) = \ln x$ which is a logarithmic function. You can check the study guide: *Integrating Basic Functions* to remind yourself of rule for integrating logarithmic functions, and so $f(x) = x \ln x - x$. You do not need the "$+c$".

2. Putting the result from step 1 and the upper and lower limits into $[f(x)]_a^b$ gives:

$$\int_1^4 \ln(x) dx = [x \ln(x) - x]_1^4$$

3. You now need to evaluate the function $f(x) = x \ln x - x$ at the lower limit of $x = 1$ and the upper limit of $x = 4$. Specifically:

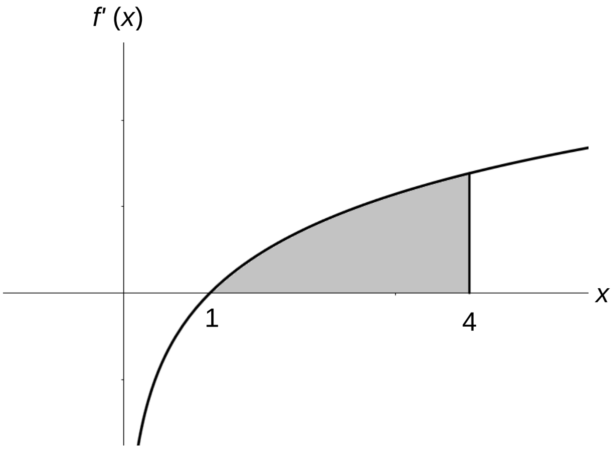
   Upper Limit: $f(4) = 4 \ln 4 - 4 = 1.55$ to 2 d.p.

   Lower Limit: $f(1) = 1 \ln 1 - 1 = -1$

4. The answer is given by $f(4) - f(1) = 1.55 - (-1) = 2.55$ to 2 d.p. So the value of the definite integral is 2.55. This can be written as:

$$\int_1^4 \ln(x) dx = [x \ln(x) - x]_1^4 = 1.55 - (-1) = 2.55 \text{ to 2 d.p.}$$

The shaded area in the figure shows the area under the graph of the function $f'(x) = \ln x$ between $x = 1$, $x = 4$ and the x-axis.



2. Here the function is $f'(x) = x^2$, the lower limit is -1 and the upper limit is 1 and so you can write the integral as:

$$\int_{-1}^{1} x^2 dx$$

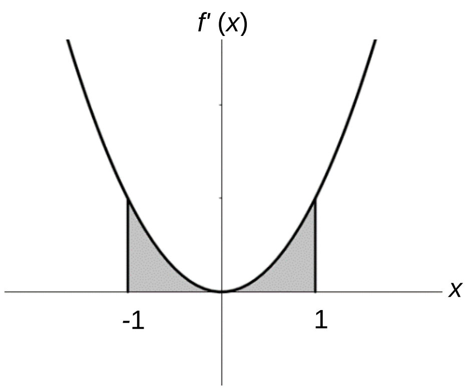
Step 1: $f'(x) = x^2$ integrates using the power rule to give $f(x) = x^3/3$

Step 2: $\int_{-1}^{1} x^2 dx = \left[\frac{x^3}{3}\right]_{-1}^{1}$

Step 3: $f(1) = \frac{1^3}{3} = \frac{1}{3}$ and $f(-1) = \frac{(-1)^3}{3} = -\frac{1}{3}$

Step 4: $\int_{-1}^{1} x^2 dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}$

So $\int_{-1}^{1} x^2 dx = \frac{2}{3}$



3. What is the area of the region enclosed by the x-axis, $f'(x) = -x^2$, and the limits $x = -1$ and $x = 1$.

Here the function is $f'(x) = -x^2$, the lower limit is -1 and the upper limit is 1 and so you can write the integral as:

$$\int_{-1}^{1} -x^2 dx$$

Step 1: $f'(x) = -x^2$ integrates using the power rule to give $f(x) = -x^3/3$

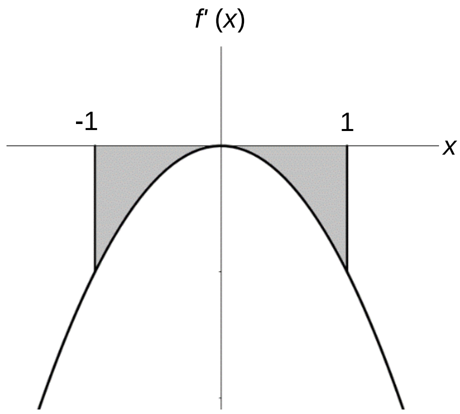
Step 2: $\int_{-1}^{1} -x^2 dx = \left[-\frac{x^3}{3}\right]_{-1}^{1}$

Step 3: $f(1) = -\frac{1^3}{3} = -\frac{1}{3}$ and $f(-1) = -\frac{(-1)^3}{3} = \frac{1}{3}$

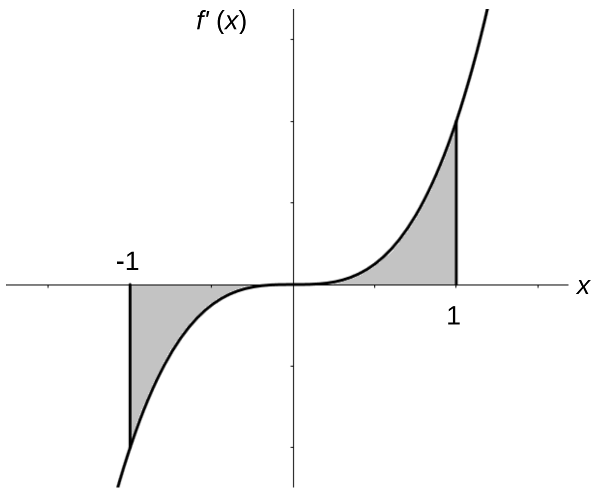
Step 4: $\int_{-1}^{1} -x^2 dx = \left[-\frac{x^3}{3}\right]_{-1}^{1} = -\frac{1}{3} - \frac{1}{3} = -\frac{2}{3}$

So $\int_{-1}^{1} x^2 dx = -\frac{2}{3}$

The definite integral has a value of $-2/3$ which is negative. As the region is entirely beneath the *x*-axis then the area is the absolute value of this which is $2/3$.



4. The area of the region enclosed by the x-axis, $f'(x) = x^3$, and the limits $x = -1$ and $x = 1$ is given in the image below:



If you directly calculate the integral of the function between the limits -1 and 1 you get:

$$\int_{-1}^{1} x^3 dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = \left(\frac{1^4}{4}\right) - \left(\frac{(-1)^4}{4}\right) = \frac{1}{4} - \frac{1}{4} = 0$$

which seems to suggest that the area under the graph is 0. However the graph above shows that is not the case. To find the area you must calculate the definite integral for the parts of the graph above and below the *x*-axis separately. The part of the function above the *x*-axis is between $x = 0$ and $x = 1$. Using these as the limits gives:

$$\int_{0}^{1} x^3 dx = \left[\frac{x^4}{4}\right]_{0}^{1} = \left(\frac{1^4}{4}\right) - \left(\frac{0^4}{4}\right) = \frac{1}{4} - 0 = \frac{1}{4}$$

The part of the graph that is below the *x*-axis is between $x = -1$ and $x = 0$ and so:

$$\int_{-1}^{0} x^3 dx = \left[\frac{x^4}{4}\right]_{-1}^{0} = \left(\frac{0^4}{4}\right) - \left(\frac{(-1)^4}{4}\right) = (0) - \left(\frac{1}{4}\right) = -\frac{1}{4}$$

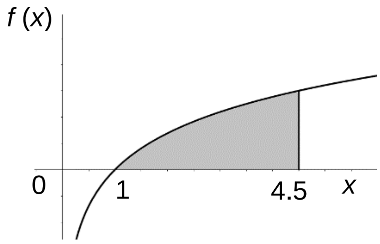
These two results show that the definite integral of the part above the *x*-axis gave $1/4$ and the definite integral of the part below gave $-1/4$. So, when the definite integral was evaluated between $x = -1$ and $x = 1$, the two results cancel each other out.

To find the area you can treat the result of the definite integral for the part of the graph wholly above the *x*-axis, as equivalent to the area of that region, $1/4$. However you need to take the absolute value for the region wholly below the *x*-axis and so the definite integral result of $-1/4$ equates to an area of $+1/4$. Adding these two results gives a total area of $1/4 + 1/4 = 1/2$.

5.

a. The definite integral $\int_1^{4.5} \ln x \, dx$ expresses the area of the shaded region of graph 2.

This piece of mathematics can be interpreted as the definite integral of the function $f'(x) = \ln x$ between the lower limit of $x = 1$ and the upper limit of $x = 4.5$.



1. The function you are integrating is $f'(x) = \ln x$ which is a logarithmic function. You can check the study guide: *Integrating Basic Functions* to remind yourself of rule for integrating logarithmic functions, and so $f(x) = x\ln x - x$. You do not need the "$+c$".

2. Putting the result from step 1 and the upper and lower limits into $[f(x)]_a^b$ gives:

$$\int_1^{4.5} \ln(x)dx = [x\ln(x) - x]_1^{4.5}$$

3. You now need to evaluate the function $f(x) = x\ln x - x$ at the lower limit of $x = 1$ and the upper limit of $x = 4.5$. Specifically:

Upper Limit: $f(4.5) = 4.5\ln 4.5 - 4.5 = 2.27$ to 2 d.p.

Lower Limit: $f(1) = 1\ln 1 - 1 = -1$

4. The answer is given by $f(4.5) - f(1) = 2.27 - (-1) = 3.27$ to 2 d.p. This can be written as:

$$\int_1^{4.5} \ln(x)dx = [x\ln(x) - x]_1^{4.5}$$

$$= f(4.5) - f(1)$$

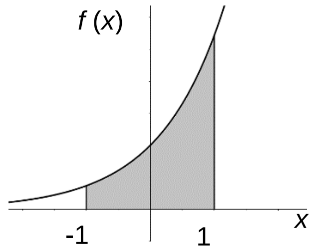
$$= (4.5\ln(4.5) - 4.5) - (1\ln(1) - 1)$$

$$= 2.27 - (-1)$$

$$= 3.27$$

b. The definite integral $\int_{-1}^{1} e^x \, dx$ expresses the area of the shaded region of graph 1.

This piece of mathematics can be interpreted as the definite integral of the function $f'(x) = e^x$ between the lower limit of $x = -1$ and the upper limit of $x = 1$.



1. The function you are integrating is $f'(x) = e^x$ this is an exponential function. You can check the study guide: *Integrating Basic Functions* to remind yourself of rule for integrating exponential functions) and so $f(x) = e^x$. You do not need the "$+c$".

2. Putting the result from step 1 and the upper and lower limits into $[f(x)]_a^b$ gives:

$$\int_{-1}^{1} e^x \, dx = [e^x]_{-1}^{1}$$

3. You now need to evaluate the function $f(x) = e^x$ at the lower limit of $x = -1$ and the upper limit of $x = 1$. Specifically:

   Upper Limit: $f(1) = e^1 = 2.72$ to 2 d.p.

   Lower Limit: $f(-1) = e^{-1} = 0.37$ to 2 d.p.

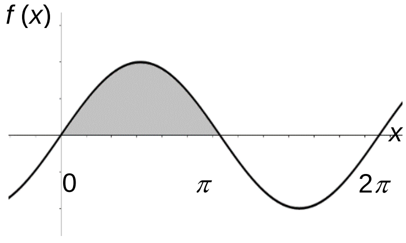
4. The answer is given by $f(1) - f(-1) = e^1 - e^{-1} = 2.72 - 0.37 = 2.35$ to 2 d.p. This can be written as:

$$\int_{-1}^{1} e^x \, dx = [e^x]_{-1}^{1} = e - e^{-1} = 2.35 \text{ to 2 d.p.}$$

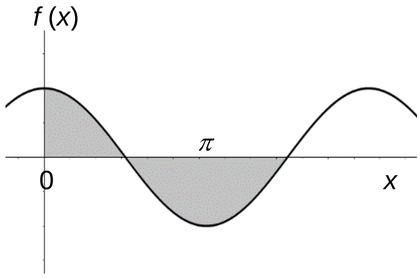
c. The definite integral $\int_0^{\pi} \sin x \, dx$ expresses the area of the shaded region of graph 3.

You can check your answer to (1e) where you have already calculated this definite integral. So:

$$\int_0^{\pi} \sin(x) dx = [-\cos x]_0^{\pi} = -\cos \pi - (-\cos 0) = 2$$



6. The graph to the right is the function $f'(x) = \cos x$.



If you directly calculate using a single definite integral of the function between the limits 0 and $3\pi/2$ you get:

$$\int_0^{3\pi/2} \cos(x)dx = [\sin(x)]_0^{3\pi/2} = \left(\sin\left(\tfrac{3\pi}{2}\right)\right) - \left(\sin(0)\right) = -1 - 0 = -1$$

To find the area you must calculate the definite integral for the parts of the graph above and below the *x*-axis separately. The part of the function above the *x*-axis is between $x = 0$ and $x = \pi/2$. Using these as the limits gives:

$$\int_0^{\frac{\pi}{2}} \cos(x)dx = [\sin(x)]_0^{\frac{3}{2}} = \left(\sin\left(\tfrac{\pi}{2}\right)\right) - \left(\sin(0)\right) = 1 - 0 = 1$$

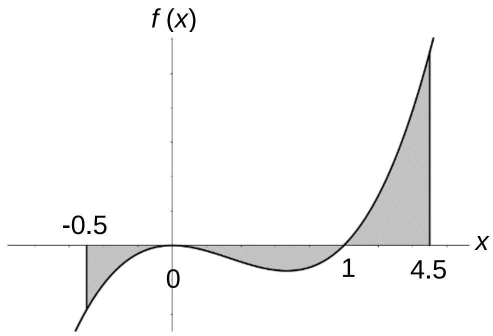
The part of the graph that is below the *x*-axis is between $x = \pi/2$ and $x = 3\pi/2$ and so:

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos(x)dx = [\sin(x)]_{\frac{\pi}{2}}^{\frac{33}{2}} = \left(\sin\left(\tfrac{3\pi}{2}\right)\right) - \left(\sin\left(\tfrac{\pi}{2}\right)\right) = -1 - 1 = -2$$

These two results show that the definite integral of the part above the *x*-axis gave 1 and the definite integral of the part below gave -2. So, when the definite integral was evaluated between $x = 0$ and $x = 3\pi/2$, the two results give a definite integral of -1.

To find the area you can treat the result of the definite integral for the part of the graph wholly above the *x*-axis, as equivalent to the area of that region, 1. However you need to take the absolute value for the region wholly below the *x*-axis and so the definite integral result of -2 equates to an area of $+2$. Adding these two results gives a total area of $1 + 2 = 3$.

7. In the graph to the right you have the function $f(x) = x^3 - x^2$.



If you directly calculate using a single definite integral of the function between the limits -0.5 and 4.5 you get:

$$\int_{-0.5}^{4.5} x^3 - x^2 dx = \left[\frac{x^4}{4} - \frac{x^3}{3}\right]_{-0.5}^{4.5}$$

$$= \left(\frac{(4.5)^4}{4} - \frac{(4.5)^3}{3}\right) - \left(\frac{(-0.5)^4}{4} - \frac{(-0.5)^3}{3}\right)$$

$$= 72.14 - 0.06$$

$$= 72.08$$

To find the area you must calculate the definite integral for the parts of the graph above and below the *x*-axis separately. The part of the function above the *x*-axis is between $x = -0.5$ and $x = 1$. Using these as the limits gives:

$$\int_{-0.5}^{1} x^3 - x^2 dx = \left[\frac{x^4}{4} - \frac{x^3}{3}\right]_{-0.5}^{1}$$

$$= \left(\frac{(1)^4}{4} - \frac{(1)^3}{3}\right) - \left(\frac{(-0.5)^4}{4} - \frac{(-0.5)^3}{3}\right)$$

$$= -0.08 - 0.06$$

$$= -0.14$$

The part of the graph that is below the *x*-axis is between $x = 1$ and $x = 4.5$ and so:

$$\int_1^{4.5} x^3 - x^2 dx = \left[\frac{x^4}{4} - \frac{x^3}{3} \right]_1^{4.5}$$

$$= \left(\frac{(4.5)^4}{4} - \frac{(4.5)^3}{3} \right) - \left(\frac{(1)^4}{4} - \frac{(1)^3}{3} \right)$$

$$= 72.14 - 0.08$$

$$= 72.06$$

These two results show that the definite integral of the part above the *x*-axis gave 72.06 and the definite integral of the part below gave -0.14. So, when the definite integral was evaluated between $x = -0.5$ and $x = 4.5$, the two results give a definite integral of 72.08

To find the area you can treat the result of the definite integral for the part of the graph wholly above the
x-axis, as equivalent to the area of that region, 72.06. However you need to take the absolute value for the region wholly below the *x*-axis and so the definite integral result of -0.14 equates to an area of 0.14. Adding these two results gives a total area of $72.06 + 0.14 = 72.20$.