Learning Enhancement Team

University of East Anglia
STUDENT SUPPORT SERVICE

Steps into Calculus

What is Integration?

This guide introduces the concept of integration of a function as an area and as the inverse of differentiation. It tells you about definite and indefinite integrals and the constant of integration.

Introduction

Calculus is one of the most important areas of mathematics. It was formalised by Sir Isaac Newton and Gottfried Leibniz in the 17th century and they, working independently, related the two major fields of calculus: **differentiation** and **integration**.

This study guide introduces the two types of integration: **definite integration** and **indefinite integration**. You will find that differentiation is taught before integration in almost all mathematical courses, this is because having a good knowledge of differentiation will help you to understand integration more easily. Therefore, before you learn integration, you should be able to differentiate basic functions comfortably (see study guides: *Differentiating Using the Power Rule* and *Differentiating Basic Functions*). Furthermore, before you learn about what integration is, you should first understand the concept of a function and its associated graph (see study guides: *Functions*, *Using Functions* and *Sketching a Graph*).

So what is integration? There are two answers:

1) Integrating a function can give the **area** between the graph of that function and the x-axis. This type of integration is called **definite integration**.

2) Integration can be thought of as the **inverse of differentiation**. In the same way that subtraction can be thought of as *undoing* addition, integration *undoes* differentiation. This type of integration is called **indefinite integration**.

This guide explains both of these ideas further, defines **definite** and **indefinite integration** and explains the **constant of integration**. The methods and rules used to integrate functions of different types are explained in the study guides: *Integrating Using the Power Rule*, *Definite Integrals*, *Integrating Basic Functions*, *Integration by Parts* and *Integration and Natural Logarithms* and in the factsheet: *Standard Integrals.*

Using integration to find an area under a graph

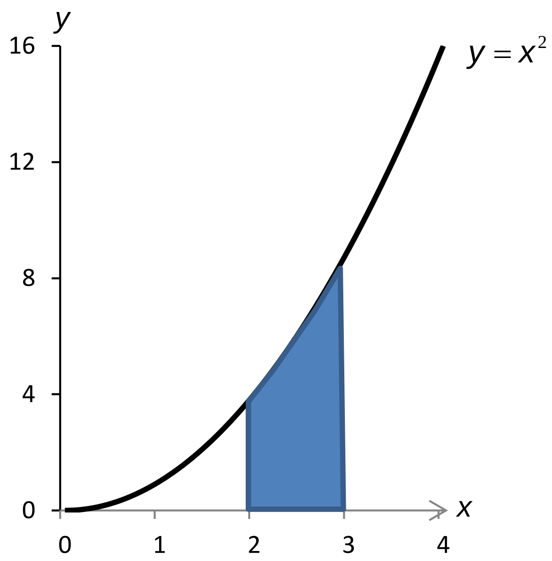
Being able to express an area as an integral and also calculating the area that an integral represents are both useful skills. In more advanced mathematics, you may have to calculate the areas and volumes of complicated curves using double or triple integrals. So practising these skills on simpler integrals first is vital.

When you integrate to find an area (or region) under a graph, you are performing **definite integration** which results in a value for the area. The value you work out is called an **integral** and is written like this:

$$\int_a^b f(x)dx$$

The meaning of these symbols is explained in the following example.

Example: Express the shaded area in the graph below as a definite integral.



The graph shows $y = x^2$ and so the shaded area is enclosed by the *x*-axis, the graph of $y = x^2$, the lower limit given by the line $x = 2$ and the upper limit given by the line $x = 3$. So the integral is written as:

$$\int_2^3 x^2 dx$$

In words this notation means "Find the area enclosed by the graph of $y = x^2$, the *x*-axis and the limits $x = 2$ and $x = 3$". Looking at the meaning of the symbols in the integral in more detail, there are four parts of this notation that you should familiarise yourself with:

1) The **integral sign** $\int$ which tells you that you are integrating.

2) The **function** which you integrating, $f(x)$. This is also the graph you are finding the area under, in the above example $f(x)=x^2$. This is known as the **integrand.**

3) The **limits** *a* and *b*. The values of *a* and *b* define the beginning and end of the region which you are trying to find the area of, moving from left-to-right. In mathematics *a* is known as the **lower limit** and *b* is known as the **upper limit**. A definite integral is called a definite integral because it is *defined* between the two limits *a* and *b*.

   They are written above and below the integral as like this $\int_a^b$.

   In this example $a=2$ and $b=3$.

4) The dx at the end. In integration dx has exactly the same meaning as dx in differentiation – an infinitely small amount of *x*. It also serves to define the variable which you are integrating **with respect to**. For example dx indicates integration with respect to the variable *x*, dt with respect to the variable *t* and so on.

> You should ensure that every integral you write has an integral sign at the beginning and (for example) dx at the end.

In general, a definite integral should always look like:

$$\int_a^b f(x)dx$$

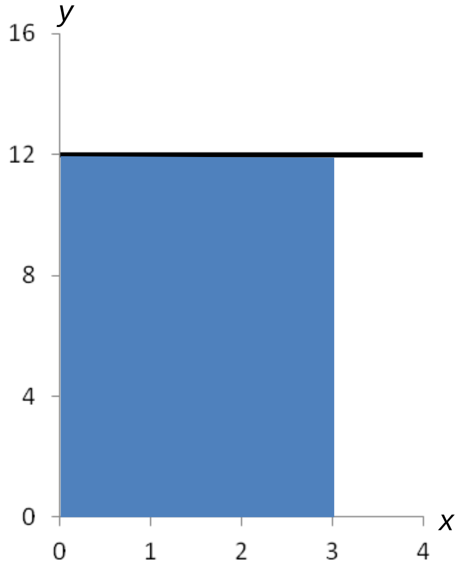
which can be interpreted to mean "Integrate the function $f(x)$ between *a* and *b* with respect to *x*" or, in more detail as "Find the area enclosed by the graph of $y=f(x)$, the *x*-axis and the limits $x=a$ and $x=b$."

Example: Integrate the function $f(x)=12$ between the limits 0 and 3.

Here the function is $f(x)=12$ and the limits are 0 and 3 and so the integral is written as:

$$\int_0^3 12dx$$

Even though this guide does not give any rules for integration, you can evaluate this definite integral using geometry. The function $f(x)=12$ is a horizontal line (see study guide: *What is a Straight Line?*) and the region described by the integration is shown as shaded in the diagram below.



As you can see the definite integral describes a rectangle of height 12 and width 3. As the area of a rectangle is found by multiplying its width by its height, the area is $3 \times 12 = 36$. In terms of the definite integral you can write:

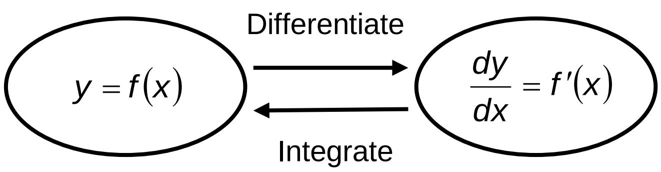
$$\int_0^3 12dx = 36$$

As this example shows, it is possible to use geometrical arguments to find the required area and exactly the same answer would be given by evaluating the integral. If a geometrical argument is not possible you will need to evaluate a definite integral (see study guide: *Definite Integrals*).

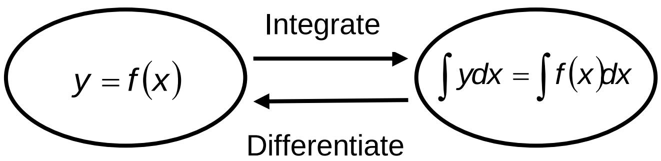
Integration as the inverse of differentiation

In the previous section you saw that definite integrals have the *x*-values *defined* by limits and result in the calculation of an area. If the limits are not included then an integral is known as **indefinite**. Integration of this type is the inverse operation of differentiation. Sometimes you will see an indefinite integral described as an **antiderivative**.

But what does this mean? Take a function $f(x)$, if you differentiate it to get $f'(x)$ and then integrate the result, you return to $f(x)$. Pictorially:



You can just as easily say that *differentiation is the inverse of integration* so, if you have a function $f(x)$, integrate it and then differentiate it, the result is the function $f(x)$.



Example: Given that the derivative of $y = x^2$ is $\frac{dy}{dx} = 2x$, what is $\int 2x dx$?

If you differentiate $y = x^2$ you get $\frac{dy}{dx} = 2x$, you can use the fact that integration is the inverse of differentiation to conclude that $2x$ **must** integrate to give x^2 and so:

$$\int 2x dx = x^2 + c$$

Notice that, when performing an indefinite integral you do not have any limits on the integral sign but there is "$+c$" at the end of your answer. The "$+c$" is called the **constant of integration** and is explained later in this guide.

When performing an indefinite integral you can always check your answer by differentiating it. If you return to the function that you were integrating to begin with then your answer is correct. Here the derivative of $y = x^2 + c$ is $\frac{dy}{dx} = 2x$, which is the function that you were integrating and so the integration is correct.

Example: What is $\int dx$?

This is a very common integral and it is useful to both know and understand the answer. Firstly the integral can be rewritten in another, more useful form as:

$$\int dx = \int 1 dx$$

This is a case where it is useful to write the hidden 1. To help find the answer you need to think of a function which differentiates to give 1. As the derivative of $y = x$ is $\frac{dy}{dx} = 1$, the integral of 1 is $x + c$ and you can write this mathematically as:

$$\int dx = \int 1 dx = x + c$$

The constant of integration

In each example of indefinite integration you may have noticed that the answer has a number *c* added to it, where *c* is a **constant** (a number that does not have an *x* associated with it). This constant *c* is known as the **constant of integration** but where does it come from?

In the first example of the previous section you were told that differentiating $y = x^2$ gives $\frac{dy}{dx} = 2x$. However, this is not the only function which differentiates to give $2x$. For example $y = x^2 + 4$ and $y = x^2 - 8$ both differentiate to give $2x$. As integration is the inverse of differentiation, if you are integrating $2x$ then the answer is a function which differentiates to give $2x$. As $y = x^2$, $y = x^2 + 4$ and $y = x^2 - 8$ all differentiate to give $2x$, which do you choose as your answer? In fact you can add (or subtract) *any* number from x^2 and its derivative will be $2x$, this is written mathematically as $y = x^2 + c$ where c represents the addition or subtraction of any number. As a direct result of this, the answer to an indefinite integral also has the constant c added to it to encompass all possible numbers which can be added or subtracted.

You can calculate the value of c, depending on the question you are answering. To do this you need extra information called **boundary conditions**. Calculations involving boundary conditions are outside the scope of this guide but a **Learning Enhancement Tutor** can help you to understand these calculations.

Summary

Definite integration gives an area between the limits on the integral sign, the curve and the *x*-axis

Indefinite integration is the opposite of differentiation and does not use limits but you need to add *c* to the end of your answer

Want to know more?

If you have any further questions about this topic you can make an appointment to see a **Learning Enhancement Tutor** in the **Student Support Service**, as well as speaking to your lecturer or adviser.

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