

Steps into Calculus

The Differential Operator

This guide introduces differentiation of a function by application of the differential operator. It then shows how this operator can be used to differentiate more complicated functions of x .

Introduction

The idea of an **operator** in mathematics is extremely useful. You can think of an operator as a piece of mathematics that does something to a **function**. Importantly, operators can describe processes in mathematics that cannot be described as functions.

One of the first examples an operator you come across is “differentiate with respect to some variable (for example x)” which is called **the differential operator**.

The differential operator with respect to x is written as either $\frac{d}{dx}$ or D_x

When you use an operator you are said to **apply** it. For instance if you apply the differential operator to a function of x you are said to be “operating **on** the function of x ”. You still use the familiar rules of differentiation to find the derivative. However, when you apply the operator, the concept of differentiation is presented in a different way. You can read the study guides: [Differentiating using the Power Rule](#), [Differentiating Basic Functions](#), [The Chain Rule](#), [The Product Rule](#) and [The Quotient Rule](#) if you need help finding derivatives.

You can also apply the differential operator to other variables, such as y , if they are a function of x . Doing so gives some notation that you are probably already familiar with:

$$\frac{d}{dx}(y) = \frac{dy}{dx}$$

when y is a function of x

For simplicity, this guide only talks about differentiation with respect to x but the notation and methods discussed are the same for other variables.

Example: Apply the differential operator to the expression $5x^4$.

When you apply the differential operator to an expression such as $5x^4$ you write:

$$\frac{d}{dx}(5x^4)$$

The result is the derivative of $5x^4$ with respect to x . This is found by using the power rule of differentiation and so:

$$\frac{d}{dx}(5x^4) = 20x^3$$

Example: Apply the differential operator to the equation $y = 5x^4$.

To do this you apply the operator to **both sides** of the equals sign and write:

$$\frac{d}{dx}(y) = \frac{d}{dx}(5x^4)$$

Here the brackets are used to imply the application of the operator, **not** multiplication. So the left-hand side shows that the operator has been **applied to** y resulting in the derivative of y with respect to x . Combining this with the result from the previous example gives:

$$\frac{dy}{dx} = 20x^3$$

which is the same as the result you would get by a direct application of the power rule.

Example: Apply the differential operator *twice* to $y = 5x^4$.

The previous example shows that after one application of the differential operator you get $dy/dx = 20x^3$. Applying the operator to both sides of this result gives:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(20x^3)$$

The left-hand side is the **second derivative** of y with respect to x which is written as:

| | |
|--|--|
| $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ | second derivative as two applications of the differential operator |
|--|--|

The notation used to write the second derivative should make more sense when seen as

a result of applying the differential operator twice on y .

Returning to the question, the right-hand side can be found using the power rule:

$$\frac{d}{dx}(20x^3) = 60x^2$$

Combining these results gives the second derivative of $y = 5x^4$:

$$\frac{d^2y}{dx^2} = 60x^2$$

The differential operator and more complicated functions

The study guide: [More Complicated Functions](#) explains that you can combine **basic functions** (power, sine, cosine, exponential and natural logarithm) in a variety of ways to make more complicated functions. If y is a function of x then you can extend this idea to include y and functions of y too as they will be composite functions of x . You use the differential operator to differentiate these kinds of functions. If y is not a function of x you need **partial differentiation** to perform differentiation (see study guide: [Basics of Partial Differentiation](#)). The following discussion assumes that y is a function of x .

Addition and subtraction

Applying the differential operator to more complicated functions made by addition and subtraction of basic functions is performed in the same way as for conventional functions, using **term-by-term differentiation**.

Example: Apply the differential operator to $y + 7x^3 = 5x^4 + 6x$.

Usually you would rearrange this function for y and then differentiate it, however you can apply the differential operator to both sides instead:

$$\frac{d}{dx}(y + 7x^3) = \frac{d}{dx}(5x^4 + 6x)$$

You apply the operator to each term in turn on each side:

$$\frac{d}{dx}(y) + \frac{d}{dx}(7x^3) = \frac{d}{dx}(5x^4) + \frac{d}{dx}(6x)$$

The first term is the derivative of y with respect to x and the other terms can be found using the power rule and so:

$$\frac{dy}{dx} + 21x^2 = 20x^3 + 6$$

You can rearrange the result to find the derivative of y with respect to x :

$$\frac{dy}{dx} = 20x^3 - 21x^2 + 6$$

Multiplication and division

Including y in the way to make more complicated functions means that you can use multiplication to make expressions such as xy or ye^x . A very useful aspect of using the differential operator is that you can apply it to these types of expressions, which are not covered by basic differentiation rules. However because y is a function of x you can still use the **product rule** to perform the differentiation.

Example: Apply the differential operator to the expression xy .

To do this notice that the term is the **product** of x and y and y is a function of x , so you use the product rule in combination with the differential operator:

$$\frac{d}{dx}(xy) = x \frac{d}{dx}(y) + y \frac{d}{dx}(x) = x \frac{dy}{dx} + y$$

The method is similar if you have an expression that can be differentiated using the **quotient rule** too.

Composition

Here, as y is a function of x , if you apply another operation to y (for example to give y^2 , $\sin y$ and so on) the result is a **composite function of x** . Therefore when you apply the differential operator you are differentiating a composite function and so you need to use the **chain rule**. You can use the chain rule to rewrite the differential operator as:

$$\frac{d}{dx} = \frac{dy}{dx} \frac{d}{dy} \quad \text{differential operator form of the chain rule}$$

Significantly, you have changed the operator from one that differentiates with respect to x (on the left) to one that differentiates with respect to y and multiplies by dy/dx (on the right).

Use the $\frac{dy}{dx} \frac{d}{dy}$ form of the differential operator to differentiate composite functions of x that are expressed as a function of y .

Example: Apply the differential operator to y^2 .

As y^2 is a composite function of x you need to use the differential operator form of the chain rule to perform the differentiation:

$$\frac{d}{dx}(y^2) = \frac{dy}{dx} \frac{d}{dy}(y^2) = 2y \frac{dy}{dx}$$

You can use the differential operator to differentiate the five basic functions, as shown in the following table.

| Function of y | Derivative |
|-----------------|-----------------------------|
| ay^n | $any^{n-1} \frac{dy}{dx}$ |
| $a \sin ky$ | $ak \cos ky \frac{dy}{dx}$ |
| $a \cos ky$ | $-ak \sin ky \frac{dy}{dx}$ |
| ae^{ky} | $ake^{ky} \frac{dy}{dx}$ |
| $a \ln(ky)$ | $\frac{a}{y} \frac{dy}{dx}$ |

You can see that the derivatives follow a similar pattern to those of the basic functions of x but, due to the application of the chain rule form of the differential operator you get multiplication by dy/dx too.

Example: Apply the differential operator to the expression $3y^3 + 7e^y + x^2 \sin y$

Let's approach this question one term at a time. Applying the differential operator to the first term gives:

$$\frac{d}{dx}(3y^3)$$

As y is cubed, this is a composite function of x and so you use the chain rule form of the differential operator:

$$\frac{d}{dx}(3y^3) = \frac{dy}{dx} \frac{d}{dy}(3y^3) = 9y^2 \frac{dy}{dx}$$

In the second term y is part of the exponential function and so you use the chain rule form of the differential operator again:

$$\frac{d}{dx}(7e^y) = \frac{dy}{dx} \frac{d}{dy}(7e^y) = 7e^y \frac{dy}{dx}$$

The final term is a product of x^2 and $\sin y$ (the latter is itself a composite function of x) and so you use the product rule in conjunction with the chain rule form of the differential operator:

$$\frac{d}{dx}(x^2 \sin y) = \sin y \frac{d}{dx}(x^2) + x^2 \frac{dy}{dx} \frac{d}{dy}(\sin y) = 2x \sin y + x^2 \cos y \frac{dy}{dx}$$

Combining these results gives:

$$\frac{d}{dx}(3y^3 + 7e^y + x^2 \sin y) = 9y^2 \frac{dy}{dx} + 7e^y \frac{dy}{dx} + 2x \sin y + x^2 \cos y \frac{dy}{dx}$$

You may notice that there are two types of terms in this result, those which are multiplied by dy/dx and those which are not. You could write:

$$\frac{d}{dx}(3y^3 + 7e^y + x^2 \sin y) = (9y^2 + 7e^y + x^2 \cos y) \frac{dy}{dx} + 2x \sin y$$

This type of factorisation is common in **implicit differentiation** (see study guide: [Implicit Differentiation](#)).



Want to know more?

If you have any further questions about this topic you can make an appointment to see a [Learning Enhancement Tutor](#) in the [Student Support Service](#), as well as speaking to your lecturer or adviser.

- 📞 Call: 01603 592761
- 💻 Ask: ask.let@uea.ac.uk
- 🔗 Click: <https://portal.uea.ac.uk/student-support-service/learning-enhancement>

There are many other resources to help you with your studies on our [website](#). For this topic, these include questions to [practise](#), [model solutions](#) and a [webcast](#).

Your comments or suggestions about our resources are very welcome.

| | | |
|---|--|--|
|  | <p>Scan the QR-code with a smartphone app for a webcast of this study guide.</p> |  |
|---|--|--|