Learning Enhancement Team

UEA University of East Anglia
STUDENT SUPPORT SERVICE

Steps into Calculus

Finding Stationary Points

This guide describes how to use the first and the second derivatives of a function to help you to locate and classify any stationary points the function may have. On the back of this guide is a flow chart which describes the process.

Introduction: Locating stationary points

The **stationary points** of a function are important in describing how that function works. Finding the stationary points of a function is a useful exercise as you can use them in sketching the function or if you need to locate where the function changes direction. The three types of stationary points are:

- a **maximum**
- a **minimum**
- a **point of inflection**

You can read the study guide: *Stationary Points* for more information about the different types (sometimes called the **nature**) of stationary points. An important point made in that study guide is that **stationary points are located where the gradient of the function is equal to zero**. As the first derivative of a function describes its gradient, you can use the **first derivative** of the function to find out information about any stationary point the function may have.

To find the location of any stationary points of a function you are looking at, use the following steps:

Step 1: Differentiate the function to find $\frac{dy}{dx}$.

Step 2: Set the derivative equal to zero and solve the equation to find value(s) for x. In other words the x-coordinates of any stationary points of your function are the solutions of the equation:

$$\boxed{\frac{dy}{dx} = 0}$$

Step 3: Substitute these values for x back into your **original** function to find the corresponding y-coordinates.

Locating stationary points requires a good working use of differentiation. You should read the study guides: *Differentiating using the Power Rule* and *Differentiating Basic Functions* to help you with this.

Example: Find the location of the stationary points of the function $y = x^3 - 3x + 2$.

Step 1: To solve this question you must first find the derivative of the function, this tells you about the gradient of the function. Using the power rule for differentiation:

$$\frac{dy}{dx} = 3x^2 - 3$$

Step 2: Set this result, $3x^2 - 3$, equal to zero and solve the quadratic equation for *x*. (If you find this difficult you can read the study guide: *Solving Quadratic Equations by Factorisation*.) Here:

$$\begin{aligned} 3x^2 - 3 &= 0 \\ 3(x^2 - 1) &= 0 \\ 3(x-1)(x+1) &= 0 \end{aligned}$$

which gives solutions of $x = -1$ and $x = 1$. This means you have two stationary points, one with an *x*-coordinate of 1 and the other with an *x*-coordinate of -1.

Step 3: You can find the corresponding *y*-coordinates by substituting $x = 1$ and $x = -1$ respectively back into the original function $y = x^3 - 3x + 2$:

at $x = 1$, $y = 1^3 - 3 \cdot 1 + 2 = 0$
so the *y*-coordinate is 0 and this gives a stationary point at $(1,0)$.

at $x = -1$, $y = (-1)^3 - 3 \cdot (-1) + 2 = 4$
so the *y*-coordinate is 4 and this gives a stationary point at $(-1,4)$.

So the function $y = x^3 - 3x + 2$ has two stationary points at $(1,0)$ and $(-1,4)$.

Classifying stationary points

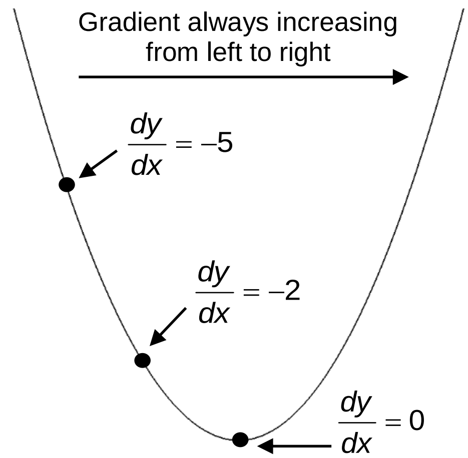
You can also use differentiation to classify the different types of stationary points. This requires **the second derivative** of the function which gives information about the rate of change of the gradient. Mathematically the **second derivative** is written as:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

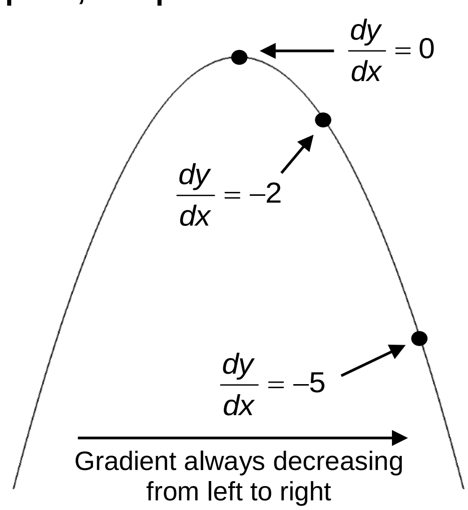
Here the left-hand side is the usual way you will see the second derivative written and the right-hand side shows that it is the derivative of the first derivative. You may also see the second derivative written as y''.

You can think of the second derivative as helping to reveal the shape of the graph of a function at some point:

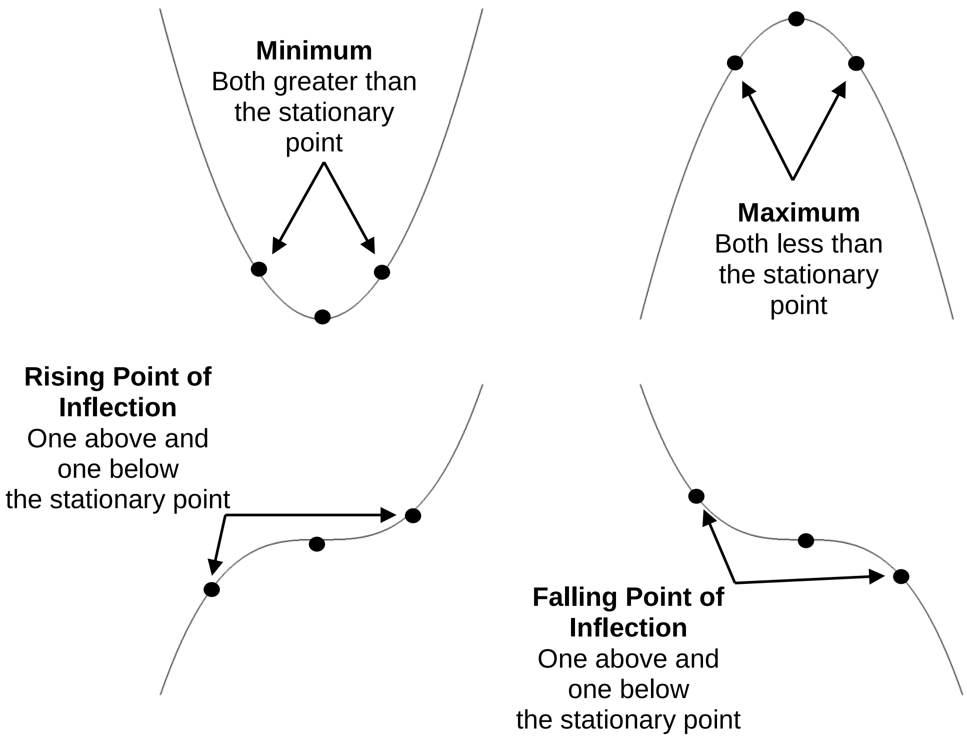
(i) **if the second derivative is positive, then the gradient of the graph is always increasing**. Look at the curve on the right which has a minimum, the gradient is always increasing as you go from left-to-right along the curve. This is obvious for uphill section, but may not be so for the downhill section. However you can think of the gradient getting less negative as you proceed along the downhill section which means the gradient is increasing (for example a gradient of -2 is more than a gradient of -5). This means that **if the second derivative is positive at a stationary point, that point is a minimum.**



(ii) **if the second derivative is negative, then the gradient of the graph is always decreasing**. Look at the curve on the right which has a maximum, the gradient is always decreasing as you go from left-to-right along the curve. Again this is obvious for positive (uphill) gradients, but may not be so for negative (downhill) gradients. You can think of the gradient getting more negative as you proceed along the downhill section which means the gradient is decreasing (a gradient of -5 is less than a gradient of -2). This means that **if the second derivative is negative at a stationary point, that point is a maximum.**



(iii) **if the second derivative is zero then the gradient of the graph is neither increasing nor decreasing**. When the second derivative is zero you cannot tell what type of stationary point you have, it could be a maximum, minimum or point of inflection. In order to classify the stationary point you need to do some further investigation. Pick two values of *x*, one either side of the stationary point, and calculate the two corresponding values of *y*. If both are more than the value of *y* at the stationary point you have a minimum, if both are less than the value of *y* at the stationary point you have a maximum and if one is more and one is less you have a point of inflection (see graphs below).



Example: Classify the stationary points of the function $y = x^3 - 3x + 2$.

Earlier in this guide you found that the stationary points of $y = x^3 - 3x + 2$ are $(1,0)$ and $(-1,4)$. To classify these points you need to find the second derivative. As the first derivative of $y = x^3 - 3x + 2$ is $3x^2 - 3$, differentiating again gives:

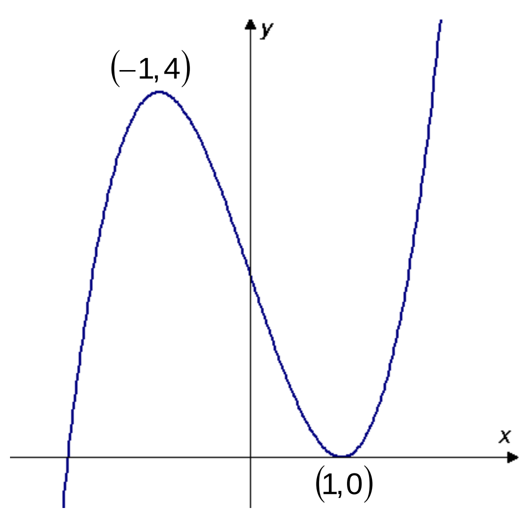
$$\frac{d^2y}{dx^2} = 6x$$

You need to work out if the value of the second derivative at each of the stationary points.

For $x = 1$ the value of the second derivative is $6 \cdot 1 = 6$ which is positive and so $(1,0)$ is a (local) minimum point.

For $x = -1$ the value of the second derivative is $6(-1) = -6$ which is negative and so $(-1,4)$ is a (local) maximum point.

This is illustrated in the graph to the right.



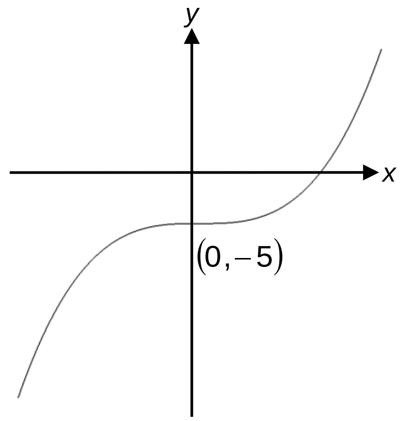
Example: Find the location and nature of any stationary points of $y=3x^3-5$.

Step 1: Using the power rule for differentiation, $\frac{dy}{dx}=9x^2$.

Step 2: Solving $9x^2=0$ for x gives a single solution of $x=0$.

Step 3: Substituting $x=0$ into $y=3x^3-5$ gives a corresponding y-coordinate of $y=-5$ and so this function has a single stationary point at $(0,-5)$.

Step 4: The second derivative of the function is $\frac{d^2y}{dx^2}=18x$.



For $x=0$ the value of the second derivative is 0 which means further investigation is needed to classify the stationary point. As $y=3x^3-5$ has a stationary point at $(0,-5)$ you need to pick values of x either side of 0, for example $x=-1$ and $x=1$ and substitute them into $y=3x^3-5$. This gives $y=-8$ and $y=-2$ respectively. As one value is larger and the other smaller than the value of y at the stationary point the function $y=3x^3-5$ has a rising point of inflection as shown by the graph to the right.

Want to know more?

If you have any further questions about this topic you can make an appointment to see a **Learning Enhancement Tutor** in the **Student Support Service**, as well as speaking to your lecturer or adviser.

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For this topic, these include questions to practise, model solutions and a webcast.

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Flowchart for finding and identifying stationary points