Learning Enhancement Team

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STUDENT SUPPORT SERVICE

Model Answers: Finding Stationary Points

Finding Stationary Points study guide

Differentiating Basic Functions study guide

Stationary Points study guide

1. To find the stationary points, this set of model answers follows the steps set out in the study guide: *Finding Stationary Points*.

(a) $y = 3x^2 - 4$ has a stationary point at $(0, -4)$.

Step 1: Differentiate the function $y = 3x^2 - 4$ using the power rule to find that $\frac{dy}{dx} = 6x$.

Step 2: Set the derivative equal to zero and solve the equation to find value for *x*:

$$6x = 0$$
$$x = 0$$

Which gives the *x*-coordinate for a single stationary point of 0.

Step 3: You now substitute this value for *x* back into the original function to find the corresponding *y*-coordinate. At $x = 0$:

$$\begin{aligned} y &= 3x^2 - 4 \\ &= 3 \cdot (0)^2 - 4 \\ &= -4 \end{aligned}$$

So the coordinate of the stationary point of $y = 3x^2 - 4$ is $(0, -4)$.

(b) $y = \frac{4}{3}x^3 - x$ has two stationary points at $\left(\frac{1}{2}, -\frac{1}{3}\right)$ and $\left(-\frac{1}{2}, \frac{1}{3}\right)$.

Step 1: Differentiate the function $y = \frac{4}{3}x^3 - x$ using the power rule to find that $\frac{dy}{dx} = 4x^2 - 1$.

Step 2: Set the derivative equal to zero and solve the equation to find value for *x*:

$$4x^2 - 1 = 0$$
$$4x^2 = 1$$
$$x = \sqrt{\tfrac{1}{4}} = \pm\tfrac{1}{2}$$

Which gives the *x*-coordinates of two stationary points of $x = \frac{1}{2}$ and $x = -\frac{1}{2}$.

Step 3: Now, in turn, substitute these values for *x* back into the original function to find the corresponding *y*-coordinates.

At $x = \frac{1}{2}$,
$$y = \tfrac{4}{3} \cdot \left(\tfrac{1}{2}\right)^3 - \tfrac{1}{2}$$
$$= -\tfrac{1}{3}$$

At $x = -\frac{1}{2}$,
$$y = \tfrac{4}{3} \cdot \left(-\tfrac{1}{2}\right)^3 - \left(-\tfrac{1}{2}\right)$$
$$= \tfrac{1}{3}$$

So the coordinates of the stationary points of $y = \frac{4}{3}x^3 - x$ are $\left(\frac{1}{2}, -\frac{1}{3}\right)$and $\left(-\frac{1}{2}, \frac{1}{3}\right)$.

(c) $y = 12x^3 - 3$ has a stationary point at $(0, -3)$.

Step 1: Differentiate the function $y = 12x^3 - 3$ using the power rule to find that $\frac{dy}{dx} = 36x^2$.

Step 2: Set the derivative equal to zero and solve the equation to find value for *x*:

$$36x^2 = 0$$
$$x = 0$$

Which gives the *x*-coordinate for a single stationary point of 0.

Step 3: You now substitute this value for *x* back into the original function to find the corresponding *y*-coordinate. At $x = 0$:

$$
\begin{aligned}
y &= 12x^3 - 3 \\
&= 12 \cdot (0)^3 - 3 \\
&= -3
\end{aligned}
$$

So the coordinate of the stationary point of $y = 12x^3 - 3$ is $(0,-3)$.

(d) $y = 4x^4$ has a stationary point at $(0,0)$.

Step 1: Differentiate the function $y = 4x^4$ using the power rule to find that $\frac{dy}{dx} = 16x^3$.

Step 2: Set the derivative equal to zero and solve the equation to find value for *x*:

$$
\begin{aligned}
16x^3 &= 0 \\
x &= 0
\end{aligned}
$$

Which gives the *x*-coordinate for a single stationary point of 0.

Step 3: You now substitute this value for *x* back into the original function to find the corresponding *y*-coordinate.

At $x = 0$,
$$
\begin{aligned}
y &= 4x^4 \\
&= 4 \cdot (0)^4
\end{aligned}
$$

So the coordinate of the stationary point of $y = 4x^4$ is $(0,0)$.

(e) $y = \frac{1}{4}x^4 - x^3 + x^2$ has three stationary points at $(0,0)$, $(1,\frac{1}{4})$, and $(2,0)$.

Step 1: Differentiate the function $y = \frac{1}{4}x^4 - x^3 + x^2$ using the power rule to find that $\frac{dy}{dx} = x^3 - 3x^2 + 2x$.

Step 2: Set the derivative equal to zero and solve the equation to find value for *x*:

$$
\begin{aligned}
x^3 - 3x^2 + 2x &= 0 \\
x(x^2 - 3x + 2) &= 0 \\
x(x-1)(x-2) &= 0
\end{aligned}
$$

Which gives the *x*-coordinates of three stationary points of $x = 0$, $x = 1$ and $x = 2$. If you find this piece of mathematics difficult you should read the study guides: *Simple Factorisation* and *Solving Quadratic Functions by Factorisation*.

Step 3: Now, in turn, substitute these values for *x* back into the original function to find the corresponding *y*-coordinates.

At $x=0$, $y=\frac{1}{4}\cdot(0)^4-0^3+0^2=0$.

At $x=1$, $y=\frac{1}{4}\cdot(1)^4-1^3+1^2=\frac{1}{4}$.

At $x=2$, $y=\frac{1}{4}\cdot(2)^4-2^3+2^2=0$.

So the coordinates of the stationary points of $y=\frac{1}{4}x^4-x^3+x^2$ are $(0,0)$, $(1,\frac{1}{4})$, and $(2,0)$.

2.

The function $y=\cos(\theta)$ has turning points at $(0,1)$, $(\pi,-1)$, and $(2\pi,1)$ in the range $0\le\theta\le 2\pi$.

Step 1: Differentiate the function $y=\cos(\theta)$ to find that $\frac{dy}{d\theta}=-\sin(\theta)$. Remember that you are differentiating with respect to θ, not *x.*

Step 2: Set the derivative equal to zero and solve the equation to find value for θ. So you have to solve:

$$-\sin(\theta)=0$$

Which asks you which values of θ (angles) have a sine of 0. This question has two extra things to consider. Firstly, when you are dealing with trigonometric functions in calculus you have to give your answers in **radians** (see study guide: *Angles*). Secondly you need to give all the angles whose sine is 0 in the range $0\le\theta\le 2\pi$. These are $\theta=0$, $\theta=\pi$ and $\theta=2\pi$, you can check these by looking at the graph of $y=\sin(\theta)$ on the factsheet: *Five Basic Functions*.

Step 3: Now, in turn, substitute these values for *x* back into the original function to find the corresponding *y*-coordinates. At $\theta=0$:

$$y=\cos(0)=1$$

At $\theta=\pi$, $y=\cos(\pi)=-1$.

At $\theta=2\pi$, $y=\cos(2\pi)=1$.

So the coordinates of the stationary points of $y=\cos(\theta)$ are $(0,1)$, $(\pi,-1)$, and $(2\pi,1)$ for the range $0\le\theta\le 2\pi$.

The function $y=2\sin(\theta)$ has turning points at $(\pi/2,2)$ and $(3\pi/2,-2)$ in the range $0\le\theta\le 2\pi$.

Step 1: Differentiate the function $y = 2\sin(\theta)$ to find that $\frac{dy}{d\theta} = 2\cos(\theta)$. Remember that you are differentiating with respect to θ, not *x.*

Step 2: Set the derivative equal to zero and solve the equation to find value for θ. So you have to solve:

$$2\cos(\theta) = 0$$

Which asks you which values of θ (angles) have a cosine of 0. This question has two extra things to consider. Firstly, when you are dealing with trigonometric functions in calculus you have to give your answers in **radians** (see study guide: *Angles*). Secondly you need to give all the angles whose cosine is 0 in the range $0 \le \theta \le 2\pi$. These are $\theta = \pi/2$ and $\theta = 3\pi/2$, you can check these by looking at the graph of $y = \cos(\theta)$ on the factsheet: *Five Basic Functions*.

Step 3: Now, in turn, substitute these values for *x* back into the original function to find the corresponding *y*-coordinates.

At $\theta = \pi/2$, $y = 2\sin(\pi/2) = 2$.

At $\theta = 3\pi/2$, $y = 2\sin(3\pi/2) = -2$.

So the coordinates of the stationary points of $y = 2\sin(\theta)$ between $0 \le x \le 2\pi$ are $(\pi/2, 2)$ and $(3\pi/2, -2)$.

3.

(a) For $y = 3x^2 - 4$, the stationary point at $(0, -4)$ is a minimum.

Step 4: Differentiate the **derivative** of the function $y = 3x^2 - 4$, $\frac{dy}{dx} = 6x$, using the power rule to find that $\frac{d^2y}{dx^2} = 6$.

As this is positive, the stationary point is a minimum.

(b) For $y = \frac{4}{3}x^3 - x$, the two stationary point at $\left(\frac{1}{2}, -\frac{1}{3}\right)$ is a minimum and $\left(-\frac{1}{2}, \frac{1}{3}\right)$ is a maximum.

Step 4: Differentiate the derivative of the function $y = \frac{4}{3}x^3 - x$, $\frac{dy}{dx} = 4x^2 - 1$, using the power rule to find that $\frac{d^2y}{dx^2} = 8x$.

You now evaluate the second derivative at each of the *x*-coordinates for the stationary points.

At $x = \frac{1}{2}$, $\frac{d^2y}{dx^2} = 8 \cdot \left(\frac{1}{2}\right) = 4$ which is positive and so $\left(\frac{1}{2}, -\frac{1}{3}\right)$ is a minimum.

At $x = -\frac{1}{2}$, $\frac{d^2y}{dx^2} = 8 \cdot \left(-\frac{1}{2}\right) = -4$ which is negative and so $\left(\frac{1}{2}, -\frac{1}{3}\right)$ is a maximum.

(c) For $y = 12x^3 - 3$ the stationary point $(0, -3)$ is a point of inflexion.

Step 4: Differentiate the derivative of the function $y = 12x^3 - 3$, $\frac{dy}{dx} = 36x^2$, using the power rule to find that $\frac{d^2y}{dx^2} = 72x$.

At $x = 0$, $\frac{d^2y}{dx^2} = 72 \cdot 0 = 0$ which means that you need to do some more investigation to determine the type of stationary point.

You choose values of *x* either side of $x = 0$ and evaluate *y* at these points to help you make the decision. Try to choose simple values for *x* so here you could use $x = -1$ and $x = 1$.

At $x = -1$, $y = 12 \cdot (-1)^3 - 3 = -15$.

At $x = 1$, $y = 12 \cdot 1^3 - 3 = 9$.

As one value of *y* is greater than the value of *y* at $x = 0$ and the other is less, the stationary point $(0, -3)$ is a point of inflexion.

(d) For $y = 4x^4$ the stationary point $(0,0)$ is a minimum.

Step 4: Differentiate the derivative of the function $y = 4x^4$, $\frac{dy}{dx} = 16x^3$, using the power rule to find that $\frac{d^2y}{dx^2} = 48x^2$.

At $x = 0$, $\frac{d^2y}{dx^2} = 48 \cdot (0)^2 = 0$ which means that you need to do some more investigation to determine the type of stationary point.

You choose values of *x* either side of where the stationary point lies, $x = 0$, and evaluate *y* at these values to help you make the decision. Try to choose simple values for *x* so here you could use $x = -1$ and $x = 1$.

At $x = -1$, $y = 4 \cdot (-1)^4 = 4$.
At $x = 1$, $y = 4 \cdot (1)^4 = 4$.

As both of these values are greater than the value of *y* at the stationary point, the stationary point $(0,0)$ is a minimum.

(e) For $y = \frac{1}{4}x^4 - x^3 + x^2$ the stationary points $(0,0)$ and $(2,0)$ are a minimums and $(1,\frac{1}{4})$ is maximum.

Step 4: Differentiate the derivative of the function $y = \frac{1}{4}x^4 - x^3 + x^2$, $\frac{dy}{dx} = x^3 - 3x^2 + 2x$, using the power rule to find that $\frac{d^2y}{dx^2} = 3x^2 - 6x + 2$.

You now evaluate the second derivative at each of the *x*-coordinates for the stationary points.

At $x = 0$, $\frac{d^2y}{dx^2} = 3 \cdot 0^2 - 6 \cdot 0 + 2 = 2$ which is positive and means that the stationary point $(0,0)$ is a minimum.

At $x = 1$, $\frac{d^2y}{dx^2} = 3 \cdot 1^2 - 6 \cdot 1 + 2 = -1$ which is negative and means that the stationary point $(1,\frac{1}{4})$ is a maximum.

At $x = 2$, $\frac{d^2y}{dx^2} = 3 \cdot 2^2 - 6 \cdot 2 + 2 = 2$ which is positive and means that the stationary point $(2,0)$ is a minimum.

(f) For $y = \cos(\theta)$, in the range $0 \le \theta \le 2\pi$, the stationary points $(0,1)$ and $(2\pi,1)$ are maximums and $(\pi,-1)$ is a minimum.

Step 4: Differentiate the derivative of the function $y = \cos(\theta)$, $\frac{dy}{dx} = -\sin(\theta)$, using the power rule to find that $\frac{d^2y}{dx^2} = -\cos(\theta)$.

You now evaluate the second derivative at each of the *x*-coordinates for the stationary points.

At $x = 0$, $\frac{d^2y}{dx^2} = -\cos(0) = -1$ which is negative and means that the stationary point $(0,1)$ is a maximum.

At $x = \pi$, $\frac{d^2y}{dx^2} = -\cos(\pi) = -(-1) = 1$ which is positive and means that the stationary point $(\pi,-1)$ is a minimum.

At $x = 2\pi$, $\frac{d^2y}{dx^2} = -\cos(2\pi) = -1$ which is negative and means that the stationary point $(2\pi,1)$ is a maximum.

(g) For $y = 2\sin(\theta)$, in the range $0 \le \theta \le 2\pi$, the stationary point $(\pi/2,2)$ is a maximum and $(3\pi/2,-2)$ is a minimum.

Step 4: Differentiate the derivative of the function $y = 2\sin(\theta)$, $\frac{dy}{dx} = 2\cos(\theta)$, using the power rule to find that $\frac{d^2y}{dx^2} = -2\sin(\theta)$.

You now evaluate the second derivative at each of the *x*-coordinates for the stationary points.

At $x = \frac{\pi}{2}$, $\frac{d^2y}{dx^2} = -2\sin\left(\frac{\pi}{2}\right) = (-2)\cdot 1 = -2$ which is negative and means that the stationary point $(\pi/2,2)$ is a maximum.

At $x = \frac{3\pi}{2}$, $\frac{d^2y}{dx^2} = -2\sin\left(\frac{3\pi}{2}\right) = (-2)\cdot(-1) = 2$ which is positive and means that the stationary point $(3\pi/2,-2)$ is a minimum.

4. Following the procedures set out above.

(a) $y = 2x - \frac{1}{6}x^3$ has a maximum $\left(2, \frac{8}{3}\right)$ at and a minimum at $\left(-2, -\frac{8}{3}\right)$.

Step 1: Differentiate the function $y = 2x - \frac{1}{6}x^3$ using the power rule to find that $\frac{dy}{dx} = 2 - \frac{1}{2}x^2$.

Step 2: Set the derivative equal to zero and solve the equation to find value for *x*:

$$2 - \tfrac{1}{2}x^2 = 0$$
$$x^2 = 4$$
$$x = \pm 2$$

Which gives the *x*-coordinates for two stationary points of 2 and -2.

Step 3: Substitute these values for *x* back into the original function to find the corresponding *y*-coordinates.

At $x = 2$, $y = 2 \cdot 2 - \frac{1}{6} \cdot 2^3 = \frac{8}{3}$.

At $x = -2$, $y = 2 \cdot (-2) - \frac{1}{6} \cdot (-2)^3 = -\frac{8}{3}$.

So the coordinates of the stationary points of $y = 2x - \frac{1}{6}x^3$ are $\left(2, \frac{8}{3}\right)$ and $\left(-2, -\frac{8}{3}\right)$.

Step 4: Differentiate the derivative from **step 2** using the power rule to find that $\frac{d^2y}{dx^2} = -x$.

Now evaluate the second derivative at each of the *x*-coordinates for the stationary points.

At $x = 2$, $\frac{d^2y}{dx^2} = -2$ which is negative and means that the stationary point $\left(2, \frac{8}{3}\right)$ is a maximum.

At $x = -2$, $\frac{d^2y}{dx^2} = -(-2) = 2$ which is positive and means that the stationary point $\left(-2, -\frac{8}{3}\right)$ is a minimum.

(b) $y = x^9$ has a point of inflexion at $(0,0)$.

Step 1: Differentiate the function $y = x^9$ using the power rule to find that $\frac{dy}{dx} = 9x^8$.

Step 2: Set the derivative equal to zero and solve the equation to find value for *x*:

$$9x^8 = 0$$
$$x = 0$$

Which gives the *x*-coordinates for a stationary point of 0.

Step 3: Substitute this value for *x* back into the original function to find the corresponding *y*-coordinate. At $x = 0$, $y = 0^9 = 0$.

So the coordinate of the stationary point of $y = x^9$ is $(0,0)$.

Step 4: Differentiate the derivative from **step 2** using the power rule to find that $\frac{d^2y}{dx^2} = 72x^7$.

At $x = 0$, $\frac{d^2y}{dx^2} = 72 \cdot (0)^2 = 0$ which means that you need to do some more investigation to determine the type of stationary point.

You choose values of *x* either side of where the stationary point lies, $x = 0$, and evaluate *y* at these values to help you make the decision. Try to choose simple values for *x* so here you could use $x = -1$ and $x = 1$.

At $x = -1$, $y = (-1)^9 = -1$.
At $x = 1$, $y = 1^9 = 1$.

As one of these values is less than the value of *y* at the stationary point, and the other is greater, the stationary point $(0,0)$ is a point of inflexion.

(c) $y = x^2 + 3x + 4$ has a minimum at $\left(-\frac{3}{2}, \frac{7}{4}\right)$.

Step 1: Differentiate the function $y = x^2 + 3x + 4$ using the power rule to find that $\frac{dy}{dx} = 2x + 3$.

Step 2: Set the derivative equal to zero and solve the equation to find value for *x*:

$$2x + 3 = 0$$
$$x = -\tfrac{3}{2}$$

Which gives the *x*-coordinates for a stationary point of $-\frac{3}{2}$.

Step 3: Substitute this value for *x* back into the original function to find the corresponding *y*-coordinate. At $x = -\frac{3}{2}$, $y = \left(-\frac{3}{2}\right)^2 + 3 \cdot \left(-\frac{3}{2}\right) + 4 = \frac{7}{4}$.

So the coordinate of the stationary point of $y = x^2 + 3x + 4$ is $\left(-\frac{3}{2}, \frac{7}{4}\right)$.

Step 4: Differentiate the derivative from **step 2** using the power rule to find that $\frac{d^2y}{dx^2} = 2$. As this is positive, the stationary point $\left(-\frac{3}{2}, \frac{7}{4}\right)$ is a minimum.

(d) $y = 10x - x^2$ has a maximum at $(5, 25)$.

Step 1: Differentiate the function $y = 10x - x^2$ using the power rule to find that $\frac{dy}{dx} = 10 - 2x$.

Step 2: Set the derivative equal to zero and solve the equation to find value for *x*:

$$10 - 2x = 0$$
$$x = 5$$

Which gives the *x*-coordinate for a stationary point of 5.

Step 3: Substitute this value for *x* back into the original function to find the corresponding *y*-coordinate. At $x = 5$, $y = 10 \cdot 5 - 5^2 = 25$.

So the coordinate of the stationary point of $y = 10x - x^2$ is $(5, 25)$.

Step 4: Differentiate the derivative from **step 2** using the power rule to find that $\frac{d^2y}{dx^2} = -2$. As this is negative, the stationary point $(5, 25)$ is a maximum.

(e) $y = 12(x+1)^3 + 6$ has a point of inflexion at $(-1, 6)$.

Step 1: Differentiate the function $y = 12(x+1)^3 + 6$ using the chain rule to find that $\frac{dy}{dx} = 36(x+1)^2$.

Step 2: Set the derivative equal to zero and solve the equation to find value for *x*:

$$36(x+1)^2 = 0$$
$$(x+1)^2 = 0$$
$$x = -1$$

Which gives the *x*-coordinate for a single stationary point of -1.

Step 3: Substitute this value for *x* back into the original function to find the corresponding *y*-coordinate. At $x = -1$, $y = 12(-1+1)^3 + 6 = 6$.

So the coordinate of the stationary point is $(-1, 6)$.

Step 4: Differentiate the derivative from **step 2** using the chain rule to find that $\frac{d^2y}{dx^2} = 72(x+1) = 72x + 72$.

Now evaluate the second derivative at the *x*-coordinates of the stationary point.

At $x = -1$, $\frac{d^2y}{dx^2} = -72 + 72 = 0$ which means that you need to do some more investigation to determine the type of stationary point.

You choose values of *x* either side of where the stationary point lies, $x = -1$, and evaluate *y* at these values to help you make the decision. Try to choose simple values for *x* so here you could use $x = -2$ and $x = 0$.

At $x = -2$, $y = 12(-2+1)^3 + 6 = -12 + 6 = -6$.
At $x = 0$, $y = 12(0+1)^3 + 6 = 12 + 6 = 18$.

As one of these values is less than the value of *y* at the stationary point, and the other is greater, the stationary point $(-1, 6)$ is a point of inflexion.

(f) The function $y = \cos(x) + x$, has a point of inflexion at $(\pi/2, \pi/2)$ in the range $0 \le x \le \pi$.

Step 1: Differentiate the function $y = \cos(x) + x$ to find that $\frac{dy}{dx} = 1 - \sin(x)$.

Step 2: Set the derivative equal to zero and solve the equation to find value for *x*:

$$1 - \sin(x) = 0$$
$$\sin(x) = 1$$

This is true in the range $0 \le x \le \pi$ only when $x = \pi/2$ (remember you have to work in radians as you have trigonometric functions).

Step 3: Substitute this value for *x* back into the original function to find the corresponding *y*-coordinate.

At $x = \frac{\pi}{2}$, $y = \cos\left(\frac{\pi}{2}\right) + \frac{\pi}{2} = 0 + \frac{\pi}{2} = \frac{\pi}{2}$.

So the coordinate of the stationary point is $(\pi/2, \pi/2)$.

Step 4: Differentiate the derivative from **step 2** to find that $\frac{d^2y}{dx^2} = -\cos(x)$.

Now evaluate the second derivative at the *x*-coordinates of the stationary point.

At $x = \pi/2$, $\frac{d^2y}{dx^2} = -\cos(\pi/2) = 0$ which means that you need to do some more investigation to determine the type of stationary point.

You choose values of *x* either side of where the stationary point lies, $x = \pi/2$, and evaluate *y* at these values to help you make the decision. Try to choose simple values for *x* so here you could use $x = 0$ and $x = \pi$ as you are working in radians.

At $x = 0$, $y = \cos(0) + 0 = 1$.
At $x = \pi$, $y = \cos(\pi) + \pi = -1 + \pi \approx 2.14$.

At the stationary point, the value of *y* is $\pi/2 \approx 1.57$. One of these values is less than 1.57 and the other is greater and so the stationary point $(\pi/2, \pi/2)$ is a point of inflexion.

5. The first derivative of the function $y = e^x$ is:

$$\frac{dy}{dx} = e^x$$

The is set to zero and solved for *x* to find the coordinates of any stationary points. In other words the *x*-coordinates of the stationary points are solution to the equation:

$$e^x = 0$$

There are no values of *x* which make $e^x = 0$. If you look at the graph of $y = e^x$ you can see that it does not cross the *x*-axis which tells you that *y* can never be 0. As $e^x = 0$ has no solution, there can be no stationary points.

There are plenty of other functions that do not have any stationary points, i.e. functions whose derivative cannot equal zero. Here is a list of some important ones to be aware of:

- $y = x^{-n} = \frac{1}{x^n}$ $\qquad \frac{dy}{dx} = -nx^{-n-1} = -\frac{n}{x^{n+1}}$.
- $y = ax$ $\qquad \frac{dy}{dx} = a$.
- $y = \ln(x)$ $\qquad \frac{dy}{dx} = \frac{1}{x}$.
- $y = a^x$ $\qquad \frac{dy}{dx} = a^x \ln a$.
- $y = \tan(x)$ $\qquad \frac{dy}{dx} = \sec^2(x)$

None of the derivatives above have values of *x* which make them equal zero.