Learning Enhancement Team

University of East Anglia
STUDENT SUPPORT SERVICE

Steps into Calculus

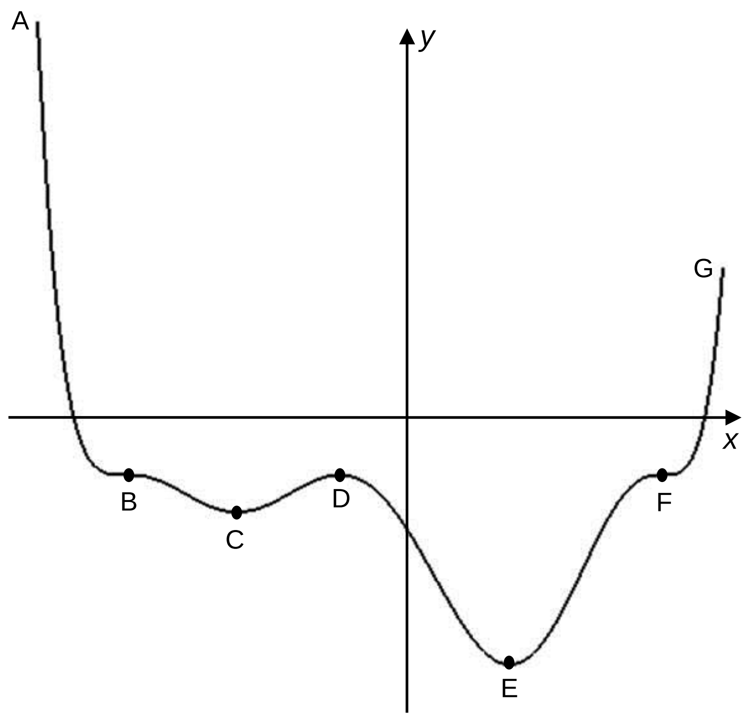
Stationary Points

This guide describes the different types of stationary points, maximums, minimums and points of inflexion, which a function may have. It also explores how the first derivative is used to locate stationary points.

Introduction

In the application of mathematics it is often extremely useful to know the point at which a **function** gives its highest or its lowest value. For example, if a **function** describes the variation of profit for a company with respect to how much of a certain good it produces, it would be useful to know the amount of good production which yields the most or **maximum** profit. Similarly, which dosage of a drug produces the smallest or **minimum** chance of contracting a particular disease? In mathematics maximum and minimum points, along with another type of point called a **point of inflexion**, are given the collective name **stationary points**. They can be found and categorised using calculus. More specifically the **first derivative** of a function is needed to locate them and **second derivative** is used to find what type they are, known as their **nature**.

A to B: Gradient is downhill and getting less steep.
B is a point of inflexion
B to C: Gradient is downhill and getting steeper.
C is a (local) minimum.
C to D: Gradient is uphill.
D is a (local) maximum.
D to E: Gradient is downhill.
E is a minimum.
E to F: Gradient is uphill and getting less steep.
F is a point of inflexion.
F to G: Gradient is uphill and getting steeper.



To get the most from this guide you should be familiar with the idea of a function and be comfortable differentiating simple functions. If you are not it is recommended that you read the study guides: *Using Functions*, *What is Differentiation?*, *Differentiation using the Power Rule*, and *Differentiating Basic Functions.* The remainder of this guide refers to the diagram on the first page.

Stationary points and gradients

Differentiating a function gives its **gradient**. Any points on the graph of a function which have a gradient equal to zero are important. Here the slope of the graph is neither uphill nor downhill. In other words these are the only points where y is not changing as x changes. The points at which the gradient of a function is zero are called the **stationary points** of the function. In the diagram on the first page of this guide the function has a gradient of zero at points B, C, D, E and F.

As the function is given by $y = f(x)$, and the gradient of the function is written as dy/dx, mathematically you may think of stationary points as solutions to the equation:

$$\frac{dy}{dx} = 0$$

where the derivative is usually also a function of x. Each individual solution of the above equation corresponds to the x-coordinate of a point on the graph of the function y. Substituting each solution in turn into the original function gives the corresponding y-coordinate. The combination of x- and y-coordinates gives the location on the function graph that corresponds to the stationary point (see study guide: *Finding Stationary Points* for details on how to carry out these calculations).

Nature of stationary points

There are a variety of points on a graph where the gradient of a function can be zero. Perhaps the most obvious points are located either where the function changes from increasing to decreasing (the corresponding function graph changes direction from uphill to downhill) or from decreasing to increasing (the graph changes from downhill to uphill); these types of points are sometimes called **turning points**. Not so obviously a graph can have a gradient of zero where the function is increasing (or decreasing) both before *and* after the point; these types of stationary points are called **points of inflexion**.

1. Maximum points

A stationary point where a function changes from increasing to decreasing and its graph changes direction from uphill to downhill as x increases is called a **maximum point**. A maximum point is also a turning point. Point D on the graph is a maximum point; before the point (from C to D) the graph is uphill and after the point (from D to E) the graph is downhill. The function itself is increasing to the left and decreasing to the right of a maximum point. Maximum points come in two forms:

(i) If the maximum point corresponds to an output that is the highest of all the outputs of the function you can say that the function has simply a **maximum** at this point. This output is **the maximum value** of the function.

(ii) If the maximum point does not correspond to the highest output of the function you can say that the function has a **local maximum** at this point. This output is a **local maximum value** of the function. Point D is a local maximum.

You can think of maximums as corresponding to the location of points on the top of any 'hills' in a graph; a maximum describes the top of the highest of these hills, if it exists. The tops of all the other hills are local maximums. A function may not have any kind of maximum points or it may have, such as the graph on the first page of this guide, one or more local maximums without having a maximum.

2. Minimum points

A stationary point in which a function changes from decreasing to increasing and its graph changes direction from downhill to uphill as x increases is called a **minimum point**. A minimum point, like a maximum point, is also a turning point. Points C and E on the graph are minimum points. The graph is downhill before each point (from B to C for point C and from D to E for point E) and uphill afterwards (from C to D for point C and from E to F for point E). The function itself is decreasing to the left and increasing to the right of a minimum point. Minimum points come in two forms:

(i) If the minimum point corresponds to an output that is the lowest of all the outputs of the function you can say that the function has simply **a minimum** at this point. This output is **the minimum value** of the function. Point E is a minimum point.

(ii) If the minimum point does not correspond to the lowest output of the function you can say that the function has a **local minimum** at this point. This output is a **local minimum value** of the function. Point C is a local minimum.

You can think of minimums as corresponding to the location of points at the bottom of any 'valleys' in a function graph; the minimum describes the bottom of the lowest of these valleys, if it exists. The bottoms of all the other valleys are local minimums. A function may not have any kind of minimum points or it may have one or more local minimums without having a minimum.

3. Points of inflexion

Points of inflexion are stationary points which are classified as neither maximums nor minimums but they do have a gradient of zero nonetheless. In these cases a function continues to increase (or decrease) near to the point but a graph with a positive gradient may level out and become horizontal before continuing uphill. An example of this can be seen at point F on the graph on page one of this guide, both before and after this point the graph has a positive gradient but at point F itself the gradient of the graph is zero.

Similarly a graph with negative gradient may level out and become horizontal before continuing downhill. An example of this can be seen at point B on the graph on page one of this guide, both before and after this point the graph has a negative gradient but at point B itself the gradient of the graph is zero.

Want to know more?

If you have any further questions about this topic you can make an appointment to see a **Learning Enhancement Tutor** in the **Student Support Service**, as well as speaking to your lecturer or adviser.

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