Learning Enhancement Team

UEA University of East Anglia

STUDENT SUPPORT SERVICE

Steps into Calculus

The Quotient Rule

This guide describes how to use the quotient rule to differentiate functions which are made by division of two basic functions. It also shows you how to use your page in an efficient way when performing lengthy calculations.

Introduction

The most important skill when differentiating is being able to identify the form of the function you have to differentiate. Once you know the form of the function you can choose an appropriate rule to differentiate it. Many complicated functions are created by combining **basic functions** in some way, usually by addition/subtraction, multiplication, division or composition (see study guide: *More Complicated Functions* and the factsheet: *Five Basic Functions*). This guide is concerned with differentiating complicated functions which are made by **dividing more basic functions**. In mathematics the word **quotient** is connected with division and is sometimes used to indicate the division of one expression by another. Given this it seems sensible that the rule used to differentiate a function which is the result of dividing simpler functions is called **the quotient rule**. The quotient rule is given by:

If	$y = \dfrac{u}{v}$
then	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

Here the function you are trying to differentiate y is made by dividing the functions u and v. Importantly, u is the **numerator** and v is the **denominator** and both are basic functions of x given in the table on the next page of this guide.

The derivative $\dfrac{dy}{dx}$ is dependent on u and v and their respective derivatives $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$.

To use the quotient rule you need to be able to differentiate basic functions. Methods to do this are discussed in the study guides: *Differentiating Using the Power Rule* and *Differentiating Basic Functions*. You should read these guides and familiarise yourself with the methods discussed in them before you continue with this guide. The results are summarised in the following table:

rule	function	derivative
1	k	0
2	ax	a
3	ax^n	anx^{n-1}
4	$a\sin kx$	$ak\cos kx$
5	$a\cos kx$	$-ak\sin kx$
6	ae^{kx}	ake^{kx}
7	$a\ln(kx)$	$\frac{a}{x}$

Throughout this guide these results will be referred to by the rule number in this table.

The quotient rule is a two-stage rule, similar to the power rule for differentiation. First you must make sure that the function you are required to differentiate fits the pattern for the quotient rule, i.e. it is made by division of two basic functions. If this is the case you can proceed to the second part which gives the derivative. The quotient rule is quite complicated and it is important that you **define *u* as the numerator and *v* as the denominator**. You also need to remember the correct order of terms in the numerator of the derivative because of the minus sign. **Sometimes it may be possible to use a simpler rule as some functions which are a quotient can be differentiated more easily using the table above or the product rule** (see study guide: *The Product Rule*). So think carefully when choosing your rule.

Example: Which of the following functions can be differentiated using the quotient rule?

(a) $y=\frac{16x^2}{\sin x}$ (b) $y=\sin\left(\frac{x}{16}\right)$ (c) $y=\frac{\sin x}{16}$ (d) $y=\frac{16\sin x}{x^2}$

(a) This function is $16x^2$ *divided by* $\sin x$ and you can differentiate it using the **quotient rule** with $u=16x^2$ and $v=\sin x$. You cannot use the product rule with $u=16x^2$ and $v=(\sin x)^{-1}$ as $v=(\sin x)^{-1}$ is not a basic function.

(b) This function is made by composition as it is the sine *of* $x/16$ so the construction of this function is complicated. In fact you can use the **chain rule** to differentiate this function (see study guide: *The Chain Rule*).

(c) Even though this function is a quotient there is no need to use the quotient rule as it can be differentiated using rule 4 in the table above with $a=1/16$ and $k=1$.

(d) This function is $16\sin x$ *divided by* x^2 and you can differentiate it using the **quotient rule**, with $u=16\sin x$ and $v=x^2$. However, you can also think of the function as $16\sin x$ *multiplied by* x^{-2} and you can use the **product rule** with $u=16\sin x$ and $v=x^{-2}$ to differentiate this function more easily (see study guide: *The Product Rule*).

Using the quotient rule

Once you have decided the function you need to differentiate is made by the division of two basics functions *and* you cannot use a simpler method to make the differentiation less difficult, you need to assign *u* and *v* as the numerator and denominator of the function respectively. When this is done you can proceed to the second part of the rule. Looking carefully:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

The derivative comprises five separate parts, each of which you have to calculate in order to find the answer. You must write the terms in the numerator in the correct order because if you swap them around your answer will have the opposite sign. As you have already defined *u* and *v* all you need to do is find their derivatives and v^2. Next you substitute these five results into the quotient rule formula and simplify if necessary. You may find it useful to divide your page vertically into an **answer space** (where your answer will be written) and an **exploring space** (where you can do any working and thinking which informs your answer). Usually in quotient rule calculations the exploring space comprises a list of the mathematics required in the substitution and any associated algebra. In more complicated mathematics the exploring space offers a vital place to play with ideas as you work towards a solution.

Example: Differentiate $y = \frac{16x^2}{\sin x}$.

As mentioned in example (a) above the function can be differentiated using the quotient rule with $u = 16x^2$ and $v = \sin x$.

Answer Space	*Exploring Space*
$y = \frac{16x^2}{\sin x}$	$u = 16x^2$
$y = \frac{u}{v}$	$v = \sin x$
So as $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$	$v^2 = \sin^2 x$
$= \frac{(\sin x)\cdot(32x) - (16x^2)\cdot(\cos x)}{\sin^2 x}$	$\frac{du}{dx} = 32x$ (rule 3)
$= \frac{16x(2\sin x - x\cos x)}{\sin^2 x}$	$\frac{dv}{dx} = \cos x$ (rule 4)

So the derivative of $y = \frac{16x^2}{\sin x}$ is $\frac{dy}{dx} = \frac{16x(2\sin x - x\cos x)}{\sin^2 x}$.

By writing *u*, *v* and so on in the exploring space to the right of the line you keep them apart from the flow of the mathematics in the answer to the left. It is also useful for further reference and checking your substitution, this is especially true when examples become more complicated.

Example: What is the gradient of $y=\dfrac{5e^x}{\cos x}$ when $x=\pi$?

The function is $5e^x$ divided by $\cos x$ so you can use the quotient rule to differentiate it by setting $u=5e^x$ and $v=\cos x$.

$$y=\frac{5e^x}{\cos x}$$

$$y=\frac{u}{v}$$

So as $$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

so $$=\frac{(\cos x)\cdot(5e^x)-(5e^x)\cdot(-\sin x)}{\cos^2 x}$$

$$=\frac{5e^x(\cos x+\sin x)}{\cos^2 x}$$

$$u=5e^x$$

$$v=\cos x$$

$$v^2=\cos^2 x$$

$\dfrac{du}{dx}=5e^x$ (rule 6)

$\dfrac{dv}{dx}=-\sin x$ (rule 5)

So the derivative of $y=\dfrac{5e^x}{\cos x}$ is $\dfrac{dy}{dx}=\dfrac{5e^x(\cos x+\sin x)}{\cos^2 x}$.

Remember that the derivative and the gradient are equivalent so when $x=\pi$ the gradient is:

$$\left.\frac{dy}{dx}\right|_{x=\pi}=\frac{5e^{\pi}(\cos\pi+\sin\pi)}{\cos^2\pi}=-5e^{\pi}=-115.703$$ to 3 decimal places.

Want to know more?

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