Learning Enhancement Team

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STUDENT SUPPORT SERVICE

Steps into Calculus

The Chain Rule

This guide describes how to use the chain rule to find the derivative of composite functions. It also shows you how to use your page in an efficient way when performing lengthy pieces of mathematics.

Introduction

The most important skill when differentiating is being able to identify the form of the function you have to differentiate. Once you know the form of the function you can choose an appropriate rule to differentiate it. Many complicated functions are created by combining **basic functions** in some way, usually by addition/subtraction, multiplication, division or composition (see study guide: *More Complicated Functions* and the factsheet: *Five Basic Functions*). This guide is concerned with differentiating complicated functions which are made by **composition of more basic functions**. The rule used to differentiate composite functions is called **the chain rule**.

To use the chain rule you need to be able to differentiate basic functions. Methods to do this are discussed in the study guides: *Differentiating Using the Power Rule* and *Differentiating Basic Functions*. You should read these guides and familiarise yourself with the methods discussed in them before you continue with this guide. The results are summarised in the following table:

rule	function	derivative
1	k	0
2	ax	a
3	ax^n	anx^{n-1}
4	$a\sin kx$	$ak\cos kx$
5	$a\cos kx$	$-ak\sin kx$
6	ae^{kx}	ake^{kx}
7	$a\ln(kx)$	$\frac{a}{x}$

Throughout this guide these results will be referred to by the rule number in this table.

If one function acts as the input to another then the output is a composite function. In turn the composite function can act as the input to another function and so on creating a

chain of functions whose output is a composite function (the longer the chain, the more complicated the composite function). The differentiation rule for composite functions is called the **chain rule**. This guide will concentrate on the simplest case of composite functions which require only one use of the chain rule. The chain rule is given by:

If	$y = f(g(x))$		
Substitute $u = g(x)$		so that	$y = f(u)$
Then	$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$		

You will sometimes see the chain rule written as $y' = g'(x)f'(u)$ which is the same but using the "*f*-dashed" notation explained in the study guide: *What is Differentiation?*.

You can find the derivative of a function using the chain rule by following these steps:

1. Check the function you are differentiating is a composite function $y = f(g(x))$.
2. Substitute $u = g(x)$ so that $y = f(u)$.
3. Differentiate $y = f(u)$ with respect to u to find $\frac{dy}{du}$.
4. Differentiate $u = g(x)$ with respect to x to find $\frac{du}{dx}$.
5. Use $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ to find the derivative.
6. Re-substitute $u = g(x)$ to replace u, so that u is not in your answer.

Example: Differentiate $y = e^{\cos x}$.

Step 1: This is a composite function where $g = \cos x$ is the input into the $u = \cos x$ exponential function. It is appropriate to use the chain rule to differentiate it.

Step 2: A suitable substitution would be the power of the exponent and so $u = \cos x$ and $y = e^u$. Both of these functions can be differentiated using the table on page 1 of this guide.

Step 3: As $y = e^u$, $\frac{dy}{du} = e^u$ (rule 6).

Step 4: As $u = \cos x$, $\frac{du}{dx} = -\sin x$ (rule 5).

Step 5: Using the results from steps 3 and 4 in the chain rule gives:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot (-\sin x)$$

Step 6: Re-substituting $u = \cos x$ gives the derivative in terms of *x* only:

$$\frac{dy}{dx} = -e^{\cos x} \sin x$$

Choosing a suitable substitution

The chain rule is an intricate piece of mathematics and requires the two-stage rule detailed above, similar to the power rule for differentiation. First you must make sure that the function you are differentiating is made by the composition of two basic functions. If this is the case, you need to use the common mathematical tool of making a **substitution**. Here you substitute a suitable basic function $g(x)$ with *u*. You can think of this mathematically as making $u = g(x)$. There are some useful methods you can use to identify a suitable substitution to make for *u*:

- use the bracketed part of functions;
  for example in $y = \sin(x^2)$ choose $u = x^2$,
  in $y = \ln(3x-2)$ choose $u = 3x-2$,
  in $y = (4-5x^7)^8$ choose $u = 4-5x^7$ and so on.
- use any mathematics under root signs;
  for example in $y = \sqrt{4-3x^2}$ choose $u = 4-3x^2$.
- use the powers of exponents; for example in $y = e^{x^2}$ choose $u = x^2$.

It is important that you can differentiate $g(x)$ with respect to *x* using the table on the first page of this guide. There is another effect of making the substitution, the original composite function $f(g(x))$ can be written as $f(u)$. If you have chosen a correct substitution you should be able to differentiate $f(u)$ with respect to *u* using the table on the first page of this guide.

Arranging your page in longer pieces of mathematics

When you are doing more complicated pieces of mathematics, such as the chain rule, you may find it useful to divide your page vertically into an **answer space** (where your answer will be written) and an **exploring space** (where you can do any working and thinking which informs your answer). In chain rule calculations the exploring space may contain a list of substitutions, the resulting derivatives and any algebra you need to use in the calculus. In more complicated mathematics the exploring space offers a vital place to play with ideas as you work towards a solution.

Using this idea, you can arrange the mathematics in the first example in this guide as:

	Answer Space	*Exploring Space*	
	$y = e^{\cos x}$		
If	$u = \cos x$ then $y = e^u$	$\frac{dy}{du} = e^u$	(rule 6)
So	$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$	$\frac{du}{dx} = -\sin x$	(rule 5)
	$= e^u \cdot (-\sin x)$		
	$= -e^u \sin x$	re-substitute *u* to give the answer in terms of *x*.	
	$= -e^{\cos x} \sin x$		

So when you differentiate $y = e^{\cos x}$ you get $\frac{dy}{dx} = -e^{\cos x} \sin x$.

Example: What is the derivative of $y = \sin(3x - 5)$?

Step 1: This is a composite function where $g(x) = 3x - 5$ is the input into the sine function. It is appropriate to use the chain rule to differentiate it.

Step 2: A suitable substitution would be the bracketed part of the sine function so $u = 3x - 5$ and $y = \sin u$. Both of these functions can be differentiated using the table on page 1 of this guide.

Step 3: As $y = \sin u$, $\frac{dy}{du} = \cos u$ (rule 4).

Step 4: As $u = 3x - 5$, $\frac{du}{dx} = 3$ (rules 1 and 2).

Step 5: Using the results from steps 3 and 4 in the chain rule gives:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (\cos u) \cdot (3) = 3\cos u$$

Step 6: Re-substituting $u = 3x - 5$ gives:

$$\frac{dy}{dx} = 3\cos(3x - 5)$$

This process can be depicted in the following way:

Answer Space		*Exploring Space*	
	$y = \sin(3x-5)$		
If	$u = 3x-5$ then $y = \sin u$	$\frac{dy}{du} = \cos u$	(rule 4)
So	$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$	$\frac{du}{dx} = 3$	(rules 1 and 2)
	$= (\cos u)\cdot(3)$		
	$= 3\cos u$	re-substitute *u* to give the answer in terms of *x*.	
	$= 3\cos(3x-5)$		

So when you differentiate $y = \sin(3x-5)$ you get $\frac{dy}{dx} = 3\cos(3x-5)$.

Example: Differentiate $y = \sqrt{5-x^2}$

Step 1: Remember that the square root is the same as raising to the power $1/2$ and so $y = \sqrt{5-x^2} = (5-x^2)^{1/2}$. This is a composite function where $g(x) = 5-x^2$ is the input into the square root. It is appropriate to use the chain rule to differentiate it.

Step 2: A suitable substitution would be the mathematics that is being square rooted so $u = 5-x^2$ and $y = u^{1/2}$. Both of these functions can be differentiated using the table on page 1 of this guide.

Step 3: As $y = u^{1/2}$, $\frac{dy}{du} = \frac{u^{-1/2}}{2} = \frac{1}{2u^{1/2}}$ (rule 3).

Step 4: As $u = 5-x^2$, $\frac{du}{dx} = -2x$ (rule 3).

Step 5: Using the results from steps 3 and 4 in the chain rule gives:

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \left(\frac{1}{2u^{1/2}}\right)\cdot(-2x) = -\frac{x}{u^{1/2}}$$

Step 6: Re-substituting $u = 5-x^2$ and replacing the root gives:

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = -\frac{x}{(5-x^2)^{1/2}} = -\frac{x}{\sqrt{5-x^2}}$$

This process can be depicted in the following way:

	Answer Space	*Exploring Space*	
	$y=\sqrt{5-x^2}$		
If	$u=5-x^2$ then $y=u^{1/2}$	$\frac{dy}{du}=\frac{u^{-1/2}}{2}=\frac{1}{2u^{1/2}}$	(rule 3)
So	$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$	$\frac{du}{dx}=-2x$	(rule 3)
	$=\left(\frac{1}{2u^{1/2}}\right)\cdot(-2x)$		
	$=-\frac{x}{u^{1/2}}$	re-substitute *u* to give the answer in terms of *x*.	
	$=-\frac{x}{\sqrt{5-x^2}}$		

So when you differentiate $y=\sqrt{5-x^2}$ you get $\frac{dy}{dx}=-\frac{x}{\sqrt{5-x^2}}$ after replacing the root.

Want to know more?

If you have any further questions about this topic you can make an appointment to see a **Learning Enhancement Tutor** in the **Student Support Service**, as well as speaking to your lecturer or adviser.

Call: 01603 592761
Ask: ask.let@uea.ac.uk
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