Learning Enhancement Team

University of East Anglia

STUDENT SUPPORT SERVICE

Steps into Calculus

Differentiating $y = ax^n$

This guide describes how to differentiate functions of the form $y = ax^n$. It identifies the various forms that these functions take and introduces the power rule of differentiation to find their derivatives.

Introduction

Many functions take the form $y = ax^n$, where n is the **power of the variable *x*** and a is called **the coefficient of *x***. Functions of this type are usually the first that you learn to differentiate. They cover a wide range of mathematics that you may have already come across. Specifically they describe straight lines, polynomials (such as quadratic and cubic functions), reciprocals and roots.

Value of *n*	Type of function	Example
0	constant	$y = 5$
1	straight line	$y = 6x$
Positive whole number greater than 1	quadratic, cubic and so on	$y = 4x^5$
negative whole number	reciprocal	$y = 1/x^3$
fraction	Containing a root	$y = \sqrt{x}$

These functions can be **differentiated** by the **power rule of differentiation** which is usually the first major rule you encounter when learning calculus. The study guide: *What is Differentiation?* can help you understand the terminology of calculus. The power rule is a two-part formula for finding the derivative of a function of the form $y = ax^n$:

If	$y = ax^n$
then	$\dfrac{dy}{dx} = anx^{n-1}$

where n is the power of x and a is the coefficient. To use the power rule successfully you need to become comfortable with how these two parts interact. To begin with, you must able to identify those functions which can be differentiated using the rule. The rule depends on you being able to identify the values of a and n from the first part of the rule and substituting them correctly into the second part.

It may be necessary to manipulate a function to see that it fits the pattern of $y = ax^n$. As differentiation naturally follows on from algebra in mathematical studies, you need to be comfortable with the many skills and techniques involved in algebra to help you with the processes and concepts of differentiation. A good understanding of the laws of indices and the rules of arithmetic involving fractions are essential – see study guides: *Multiplying and Dividing Fractions*, *Cancelling Down Fractions*, *Multiplying and Dividing Algebraic Fractions* and *Laws of Indices*. You should also be familiar with the concept of a function (see study guides: *Functions* and *Using Functions*).

Using the power rule

The rest of this guide contains examples of the variety of functions which can be differentiated using the power rule. The examples will cover a variety of different cases for *a* and *n* and are chosen to highlight common difficulties.

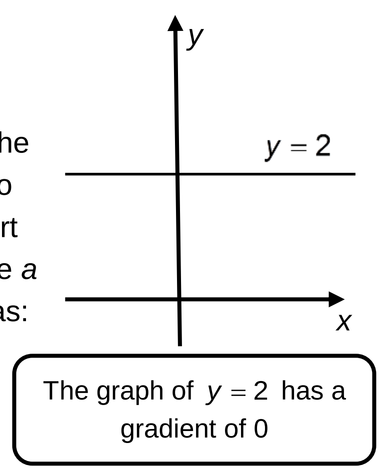
1. Horizontal lines (see study guide: *What is a Straight Line?*)

Horizontal lines are represented by the equation $y = a$. This function does not look like $y = ax^n$ at the moment and so you may think that you cannot use the power rule to differentiate it. However the laws of indices tells you that $x^0 = 1$ and so as:

$y = a = a \cdot 1$ replacing the 1 with x^0 gives $y = ax^0$

Where the dot implies multiplication. By thinking about the graph that the equation represents, you may gain some insight into its derivative. Remember that the gradient of a graph is given by its derivative and by the change in *y* divided by the change in *x.* For a horizontal line, the change in *y* is zero and so the gradient of a horizontal line is also zero. This is an important result that is reflected in the fact that **the derivative of a constant is zero**.

Example: Differentiate $y = 2$.



As $y = 2$ is a horizontal line you already know that its gradient is zero. If you can correctly identify *a* and *n* in the first part of the power rule you can use the second part to differentiate the function. It may help to write the first part of the power rule underneath the function to help you see *a* and *n*. Using the laws of indices you can rewrite $y = 2$ as:

$$y = 2x^0$$
$$y = ax^n$$

From the pattern of the first part you can identify $a = 2$ and $n = 0$, remember that *n* is the power of *x* and *a* is the coefficient of *x*. For horizontal graphs *a* is always the constant in the function. As you know *a* and *n* you can use the second part of the power rule to find the derivative:

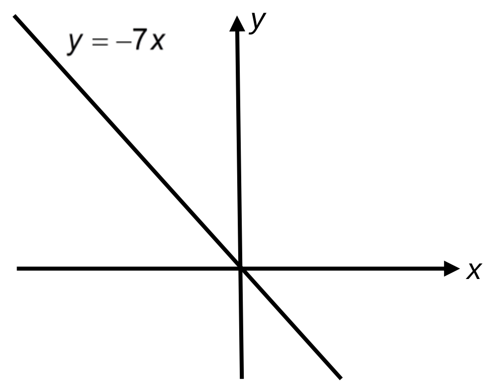
$$\frac{dy}{dx} = anx^{n-1} = 2 \cdot 0 \cdot x^{0-1} = 0$$ and so $y = 2$ has a gradient of 0.

2. Straight lines of the form $y = ax$ (see study guide: *What is a Straight Line?*)

Functions of the form $y = ax^n$ with $n = 1$ can be written as $y = ax$. These functions represent straight lines with gradient *a* which pass through the origin. As the gradient of any line is given by its derivative, the derivative of $y = ax$ must equal *a*. So, intuitively, the value of *a* in a function of this type should give the same result as the power rule of differentiation.

Example: What is the gradient of $y = -7x$?

As $y = -7x$ is a straight line through the origin (see graph below), its gradient is given by the coefficient of *x* which is -7. You can use the laws of indices to rewrite $y = -7x$ as $y = -7x^1$ and so use power rule:



$$y = -7x$$
$$y = -7x^1$$

This gives $a = -7$ and $n = 1$. As your function fits the pattern for the first part of the power rule, you can use the second part to find the derivative:

$$\frac{dy}{dx} = anx^{n-1} = -7 \cdot 1 \cdot x^{1-1} = -7x^0 = -7$$

So $y = -7x$ has a derivative of -7, which is the same as the expected gradient.

Covering up method: A useful method to use to find *a* in the power rule is to cover up *x* and its power, what remains is the value of *a*. For example in $y = -7x$ if you cover up the *x* you have -7 remaining, which is the correct value of *a*.

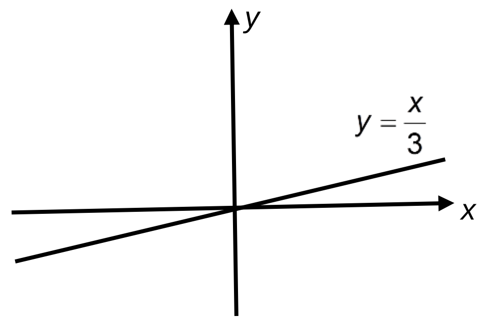
Example: Differentiate $y = \frac{x}{3}$.

It is not as easy to identify *a* in this example. A common error is to say that *a* is 3. However, you should be careful when your function is a fraction. By using your knowledge of how fractions multiply together, as well as the laws of indices, you can rewrite the function as:

$$y = \frac{x}{3} = \frac{1}{3} \cdot \frac{x}{1} = \frac{1}{3} x^1$$

So this gives $a = 1/3$ and $n = 1$, you can use these values in the second part of the power rule:

$$\frac{dy}{dx} = anx^{n-1} = \frac{1}{3} \cdot 1 \cdot x^{1-1} = \frac{1}{3} x^0 = \frac{1}{3}$$



The function is a straight line, with gradient of $1/3$, which passes through the origin, as shown in the graph above.

3. Positive, whole powers of *x*

A very common type of function to be required to differentiate has *n* as a positive whole number.

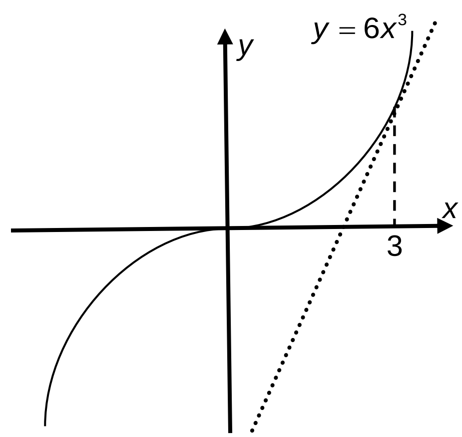
Example: Differentiate $y = 6x^3$ and then find its gradient at $x = 3$.

Here the function obviously fits the first part of the power rule:

$$y = 6x^3$$
$$y = ax^n$$

With $a = 6$ and $n = 3$ (you can cover up x^3 to see that $a = 6$). Using these values in the second part of the power rule gives:

$$\frac{dy}{dx} = anx^{n-1} = 6 \cdot 3 \cdot x^{3-1} = 18x^2$$



The derivative $18x^2$ describes the gradient of $y = 6x^3$ at every point on its graph. You should realise that the gradient of $y = 6x^3$ changes as *x* changes as its derivative is a function of *x*. To find the gradient when $x = 3$, or indeed at any point, you substitute the value into the derivative function:

$$\frac{dy}{dx} = 18x^2 = 18 \cdot 3^2 = 162$$

On the graph above, the dotted line illustrates the tangent on the curve when $x = 3$, calculus confirms that the gradient of this line is 162.

Example (with the coefficient hidden): Differentiate $y = x^4$.

In order to use the power rule you need to rewrite the function to fit the pattern because the coefficient *a* seems to be missing from $y = x^4$. Remember you can always multiply something by 1 and keep it the same so the function can be rewritten as $y = 1 \cdot x^4$. Often, when the coefficient is absent, it is assumed that its value is zero which is not the case. If it *were* zero then the whole function would be zero as anything multiplied by zero is zero. However, using $y = 1 \cdot x^4$ you can use the first part of the power rule:

$$y = 1 \cdot x^4$$
$$y = ax^n$$

Which gives $a = 1$ and $n = 4$. You can use these values in the second part of the power rule to find the derivative:

$$\frac{dy}{dx} = anx^{n-1} = 1 \cdot 4 \cdot x^{4-1} = 4x^3$$

So the derivative of $y = x^4$ is $4x^3$.

Example (with fractional coefficient): Differentiate $y = \frac{5x^3}{6}$.

Again this example does not perfectly fit the pattern for the power rule and you should take care when deciding the value of *a*. You can cover up x^3 to find that the value of *a* is $5/6$ or you can rewrite the function as:

$$y = \frac{5x^3}{6} = \frac{5}{6} \cdot \frac{x^3}{1} = \frac{5}{6}x^3$$

showing that $a = 5/6$ and $n = 3$. You can use these values in the second part of the power rule to find the derivative:

$$\frac{dy}{dx} = anx^{n-1} = \frac{5}{6} \cdot 3 \cdot x^{3-1} = \frac{5}{2}x^2 = \frac{5x^2}{2}$$

where the derivative function has been cancelled down to give the simplest form.

4. Negative, whole powers of *x*: reciprocals

When *x* appears beneath the dividing line it is extremely common to assign an incorrect value for *n*. However, by recalling the law of indices $1/x^m = x^{-m}$, for non-zero *m*, helps you rewrite these functions in a form which matches the first part of the power rule.

Example: Differentiate $y = \frac{3}{x^5}$.

You may think that *n* is 5 in this question, but by using the law of indices mentioned above you can see that:

$$y = \frac{3}{x^5} = \frac{3}{1} \cdot \frac{1}{x^5} = 3x^{-5}$$

which fits the pattern of the first part of the power rule with $a = 3$ and $n = -5$. You can use these values in the second part of the power rule to find the derivative:

$$\frac{dy}{dx} = anx^{n-1} = 3 \cdot (-5) \cdot x^{-5-1} = -15x^{-6} = -\frac{15}{x^6}$$

Where the derivative function is written in the form of the question, with *x* beneath the dividing line.

5. Powers of *x* which are unit fractions: roots

You can use the law of indices $\sqrt[m]{x} = x^{1/m}$ to help you to differentiate functions which contain a root.

Example: Differentiate $y = 8\sqrt[2]{x}$.

By using the law of indices mentioned above you can see that:

$$y = 8\sqrt[2]{x} = 8x^{1/2}$$

which fits the pattern of the first part of the power rule with $a = 8$ and $n = 1/2$. You can use these values in the second part of the power rule to find the derivative:

$$\frac{dy}{dx} = anx^{n-1} = 8 \cdot \frac{1}{2} \cdot x^{(1/2)-1} = 4x^{-1/2} = \frac{4}{x^{1/2}} = \frac{4}{\sqrt[2]{x}}$$

where the derivative function is written in the form of the question, as a root.

Want to know more?

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