University of East Anglia
STUDENT SUPPORT SERVICE

Steps into Discrete Mathematics

Basics of Relations

This guide introduces the basics of mathematical relations as the Cartesian product of two sets. It explains the language used to describe them. It also defines special types of relations called reflexive, symmetric, anti-symmetric and transitive and equivalence.

Introduction

A mathematical **relation** describes the result of choosing elements from a **set** or sets **in a specific order**. This guide is concerned with relations which result from selecting elements either from **two different sets** or **two from one set**. Technically you would call this a **binary relation** but simply "relation" is far more common. As relations are closely related to sets, to make the most of this guide you should familiarise yourself with the common language and symbols of sets (see study guide: *Basics of Sets*).

If you have two sets, a set of mammals $M = \{\text{lion, cat, rat}\}$ and a set of words used to classify them $C = \{\text{feline, rodent}\}$, all the ways of choosing an element from set M first and then pairing it with an element from set C second are:

(lion, feline), (lion, rodent), (cat, feline), (cat, rodent), (rat, feline), (rat, rodent)

In mathematics elements chosen in this way are called an **ordered pair** as there are two things in a particular order: firstly an element from set M and then an element from set C. For example:

(lion, rodent)	**representation of an ordered pair**

Notice that the ordered pair are written in a similar way to coordinates. In fact coordinates in the xy–plane are examples of ordered pairs.

This list above forms a set called the **Cartesian product** of M and C and is written $M \times C$ (see study guide: *Operations on Sets*). A relation always describes a **subset** of $M \times C$ and is said to be **from *M* to *C***. For two general sets A and B:

> Any relation R **from** a set A **to** a set B is always a **subset** of $A \times B$.

Often the letter R is used to depict a relation. The elements of set A are called the **range** of the relation and the elements of set B are called the **domain** of the relation.

Sometimes the ordered pairs in a relation make sense, for example you could describe a relation from M to C given by:

$$R = \{(\text{lion, feline}), (\text{cat, feline}), (\text{rat, rodent})\}$$

where you can think of the relation in terms of:

(element of) *M* **is a** (element of) *C*

Similarly you could describe a relation from M to C given by:

$$R = \{(\text{rat, rodent})\}$$

which can be thought of as:

(element of) *M* **starts with the same letter as** (element of) *C*

However, *any* subset of $M \times C$ you choose forms a relation and:

$$R = \{(\text{cat, rodent}), (\text{lion, rodent}), (\text{rat, feline})\}$$

is still a valid relation but the ordered pairs do not make any sense in terms of an explanation in language. However:

$$R = \{(\text{cat, rodent}), (\text{lion, cat}), (\text{rodent, feline})\}$$

is *not* a relation from M to C as (lion, cat) and (rodent, feline) are not elements of $M \times C$.

Special categories of relations

From now on this guide only considers special relations that are common in the study of mathematics. All these special relations are subsets of the Cartesian product $A \times A$, that is the first *and* second elements are taken **from the same set**. This may seem strange but if you think of relations concerning numbers, you would expect that the numbers would both come from the same set (integers or real numbers for example). For more details about the different kinds of numbers you can read the study guide: *Different Kinds of Numbers*.

1. Reflexive relations

In order for a relation R on a set A to be **reflexive**, it must contain all the ordered pairs where an element is paired with itself. Mathematically speaking it must contain (a, a) where a is **any element** of A. In other words:

Reflexive relation: Contains every ordered pair (a,a) for all *a* in the set *A*.

This definition is best illustrated by an example.

Example: Which of the following relations on the set $M = \{\text{lion, cat, rat}\}$ are reflexive?

Here the elements are lion, cat, and rat and so, for a relation to be reflexive it must contain (lion, lion), (cat,cat) **and** (rat, rat), **it does not matter what else it contains**.

(i) $R = \{(\text{lion, lion}), (\text{rat, rat}), (\text{cat,cat})\}$
is **reflexive** as it contains (a,a) for all *a*. (This relation describes "is a".)

(ii) $R = \{(\text{lion, lion}), (\text{rat, rat}), (\text{cat,rat}), (\text{cat,cat})\}$
is **reflexive** as it contains (a,a) for all *a*.

(iii) $R = \{(\text{lion, lion}), (\text{cat,rat}), (\text{cat,cat})\}$
is **not reflexive** as it does not contain (rat, rat).

Symbols such as $\leq$, $\geq$, $\subseteq$ and $=$ are used to describe relations which are reflexive.

2. Symmetric relations

A **symmetric** relation is a relation in which every element is paired with another element which has the chosen elements reversed. In other words:

Symmetric relation: Contains elements (a,b) **and also** (b,a),
for two elements *a*, *b* chosen from set *A*.
Elements (a,a) are always symmetric.

You can think of (a,b) and (b,a) as being **symmetric partners** of each other. Elements (a,a) are symmetric anyway and you can think of them as symmetric partners of themselves. A good technique here is to look for elements that **do not** have a symmetric partner. If you find one then the relation is **not** symmetric. Otherwise, if all the elements have symmetric partners, then your relation is symmetric.

Example: Which of the following relations on the set $M = \{\text{lion, cat, rat}\}$ are symmetric?

(i) $R = \{(\text{lion, lion}), (\text{rat, rat}), (\text{cat,cat})\}$
is **symmetric** as all the elements are their own symmetric partner.
(Note that this relation is also reflexive.)

(ii) $R = \{(\text{lion, lion}), (\text{rat, rat}), (\text{cat,rat}), (\text{cat,cat})\}$
is **not symmetric** as (cat, rat) is present but not its symmetric partner (rat,cat). (Note that this relation is reflexive.)

(iii) $R = \{(\text{rat, rat}), (\text{cat,rat}), (\text{rat,cat})\}$
is **symmetric** as all the elements have a symmetric partner present.

3. Antisymmetric relations

You can use the idea of a symmetric partner, introduced for symmetric relations, to help you identify **antisymmetric** relations too. It is important to note that antisymmetric does not mean the same thing as not symmetric. Indeed there are many relations which are both symmetric and antisymmetric.

Antisymmetric relation: Contains no examples of the symmetric partners (a,b) and (b,a) (where *a* and *b* are different). However elements (a,a) are allowed in antisymmetric relations.

A good technique here is to look for any elements in your relation that have a symmetric partner. If you find such a pairing then the relation is **not** antisymmetric. Otherwise, your relation is antisymmetric.

Example: Which of the following relations on the set $M = \{\text{lion, cat, rat}\}$ are antisymmetric?

(i) $R = \{(\text{lion, lion}), (\text{rat, rat}), (\text{cat,cat})\}$
is **antisymmetric** as no examples of the symmetric partners (a,b) and (b,a) are present, all the symmetric partners are of the type (a,a).
(Note that this relation is also reflexive and symmetric.)

(ii) $R = \{(\text{lion, lion}), (\text{rat, rat}), (\text{cat,rat}), (\text{cat,cat})\}$
is **antisymmetric** as no examples of the symmetric partners (a,b) and (b,a) are present, all the symmetric partners are (a,a).
(Note that this relation is reflexive but not symmetric.)

(iii) $R = \{(\text{rat, rat}), (\text{cat,rat}), (\text{rat,cat})\}$
is not **antisymmetric** as the symmetric partners (cat,rat) and (rat,cat) are present. (Note this relation is symmetric.)

(iv) $R = \{(\text{rat, cat}), (\text{lion, cat})\}$

is **antisymmetric** as no examples of the symmetric partners (a,b) and (b,a) are present.

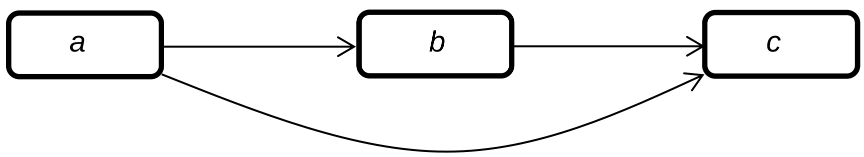
4. Transitive relations

Identifying a **transitive** relation can be a bit more difficult than the other three types of relation. It involves three elements *a*, *b* and *c* from the set *A*.

Transitive relation: If it contains (a,b) and (b,c) then it must also contain (a,c).

You can think of a transitive relation this way: if the relation contains elements which describe a journey with a beginning, an end and a stop in the middle it must also contain the direct journey from beginning to end. So if (a,b) is represented by $\boldsymbol{a \to b}$ and (b,c) is represented by $\boldsymbol{b \to c}$ then a transitive relation must also have (a,c) $\boldsymbol{a \to c}$ too.

Pictorially:

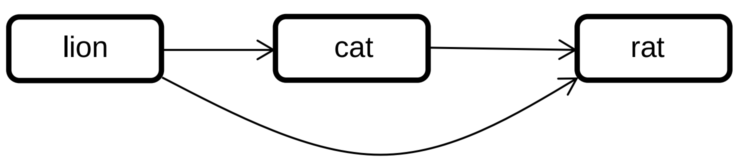


The elements (a,a) and (a,b), and also (a,b) and (b,b), are transitive as you are always able to go directly from $\boldsymbol{a \to b}$.

Example: Which of the following relations on the set $M = \{\text{lion, cat, rat}\}$ are transitive?

(i) $R = \{(\text{lion, cat}), (\text{cat, rat}), (\text{lion, rat})\}$

is **transitive** as lion $\to$ cat $\to$ rat but also lion $\to$ rat. You could draw:



This relation could describe "eats" as lion eats cat which eats rat *and* lion eats rat.

(ii) $R = \{(\text{lion, lion}), (\text{cat, rat}), (\text{cat, cat}), (\text{rat,rat})\}$

is **transitive** as (a) cat $\to$ rat $\to$ rat and cat $\to$ rat,
(b) cat $\to$ cat $\to$ rat and cat $\to$ rat.

(iii) $R = \{(\text{lion, lion}), (\text{lion, rat}), (\text{rat,cat})\}$

is not **transitive** as lion $\to$ rat $\to$ cat but there is no lion $\to$ cat.

Equivalence relations

> If a relation is **reflexive, symmetric and transitive** then it is called an **equivalence relation**.

Some fundamental mathematical ideas such as “is equal to” and (for triangles) “is congruent to” are equivalence relations.

Example: Is the relation $R = \{(\text{lion, lion}), (\text{cat, rat}), (\text{cat, cat}), (\text{rat,rat}), (\text{rat, cat})\}$ on the set $M = \{\text{lion, cat, rat}\}$ an equivalence relation?

Is *R* reflexive? **Yes** as it contains (lion, lion), (cat,cat) and (rat,rat).

Is *R* symmetric? **Yes** as each element has a symmetric partner in the set.

Is *R* transitive? **Yes** it is **transitive** as:

(i) $\text{cat} \to \text{rat} \to \text{rat}$ *and* $\text{cat} \to \text{rat}$, (ii) $\text{cat} \to \text{cat} \to \text{rat}$ *and* $\text{cat} \to \text{rat}$,

(iii) $\text{cat} \to \text{rat} \to \text{cat}$ *and* $\text{cat} \to \text{cat}$ (iv) $\text{rat} \to \text{cat} \to \text{cat}$ *and* $\text{rat} \to \text{cat}$,

(v) $\text{rat} \to \text{cat} \to \text{rat}$ *and* $\text{rat} \to \text{rat}$ (vi) $\text{rat} \to \text{rat} \to \text{cat}$ *and* $\text{rat} \to \text{cat}$

So the relation $R = \{(\text{lion, lion}), (\text{cat, rat}), (\text{cat, cat}), (\text{rat,rat}), (\text{rat, cat})\}$ is an equivalence relation on *M*.

Want to know more?

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