Learning Enhancement Team

University of East Anglia
STUDENT SUPPORT SERVICE

Model answers: Basics of Relations

Basics of Relations study guide

Remember that a relation from A to B is a subset of the Cartesian product $A \times B$.
Remember that a relation R is:

- reflexive if (x,x) is in R for all x in the set A;
- symmetric if whenever (x,y) is in R then (y,x) is also in R;
- antisymmetric if no examples of the symmetric partners (x,y) and (y,x) exist in R (where x and y are different)
- transitive if when R contains (x,y) and (y,z) then R contains (x,z) as well.

1. a) $(1) = \{(\text{triangle, green}), (\text{circle,red}), (\text{cross,blue}), (\text{square,pink})\}$ is a relation from A to B.

You can see that each element of (1) is an element of $A \times B$ and so (1) is a subset of $A \times B$. So you can say that (1) is a relation from A to B.

b) $(2) = \{(\text{up,square}), (\text{left,cross}), (\text{pink, blue})\}$ is not a relation from C to A.

You can see that the element (pink, blue) of (2) is not an element of $C \times A$ as blue is not an element of C. So you can say that (2) is not a relation from C to A.

c) $(3) = \{(\text{up, green}), (\text{down,blue}), (\text{left,pink}), (\text{red,right})\}$ is not a relation from B to C.

You can see that the element (red,right) of (3) is not an element of $B \times C$ as red is not an element of B. So you can say that (3) is not a relation from B to C.

d) $(4) = \{(\text{triangle, left})\}$ is a relation from A to C.

You can see that each element of (4) is an element of $A \times C$ and so (4) is a subset of $A \times C$. So you can say that (4) is a relation from A to C.

2. $(5)=\{(a,b),(b,a)\}$ is symmetric, but not reflexive, antisymmetric or transitive.

(5) is not reflexive as you can see that it does not contain the element (a,a) of $D \times D$.

(5) is symmetric as you can see it contains the symmetric partners (a,b) and (b,a) and these are the only elements of (5).

(5) is not antisymmetric as you have seen that it contains the symmetric partners (a,b) and (b,a), where a and b are different.

(5) is not transitive. You can see that (5) contains (a,b) and (b,a) but it does not contain (a,a).

$(6)=\{(b,c),(c,d),(b,d)\}$ is antisymmetric and transitive, but not reflexive or symmetric.

(6) is not reflexive as it does not contain the element (a,a) of $D \times D$.

(6) is not symmetric as you can see that (6) contains the element (b,c) but it does not contain its symmetric partner (c,b).

(6) is antisymmetric, notice that there are no pairs of symmetric partners in (6).

(6) is transitive. You can see this as (6) contains (b,c), (c,d) it also contains (b,d). There are no other pairs like this to check in (6) and so it is transitive.

$(7)=\{(a,a),(a,b),(b,c),(a,c),(b,b),(c,d),(b,d)\}$ is antisymmetric, but not reflexive, symmetric or transitive.

(7) is not reflexive because you it does not contain the element (c,c) of $D \times D$.

(7) is not symmetric as you can see that (7) contains the element (c,d) but it does not contain its symmetric partner (d,c).

(7) is antisymmetric as you can notice that there are no pairs of symmetric partners in (7). Although we have that (a,a) is in (7), you can remember from the study guide that these pairs are acceptable in antisymmetric relations.

(7) is not transitive. You can see that (7) contains (a,c) and (c,d) but it does not contain (a,d).

$(8)=\{(a,a),(b,b),(a,c),(c,a),(c,d),(c,c),(d,d)\}$ is reflexive, but not symmetric, antisymmetric or transitive.

(8) is reflexive as you can see that it contains (a,a), (b,b), (c,c) and (d,d); and so (8) contains (x,x) for all x in the set D.

(8) is not symmetric as it contains (c,d) but not its symmetric partner (d,c).

(8) is not antisymmetric as you have seen that it contains the symmetric partners (a,c) and (c,a), where a and c are different.

(8) is not transitive. You can see that (5) contains (a,c) and (c,d) but it does not contain (c,d).

$(9)=\{(a,a),(b,b),(a,b),(b,a),(c,d),(d,c),(c,c)\}$ is symmetric, but not reflexive, antisymmetric or transitive.

(9) is not reflexive as you can see that it does not contain the element (d,d) of $D\times D$.

(9) is symmetric as you can see it contains the symmetric partners (a,b) and (b,a), as well as (c,d) and (d,c). There are no more elements in (9) that do not have symmetric partners, and so you can say that (9) is symmetric.

(9) is not antisymmetric as you have seen that it contains the symmetric partners (a,b) and (b,a), where a and b are different.

(9) is not transitive. You can see that (9) contains (d,c) and (c,d) but it does not contain (d,d).

$(10)=\{(a,b),(b,c),(c,d),(a,c),(b,d)\}$ is antisymmetric, but not reflexive, symmetric or transitive.

(10) is not reflexive because it does not contain the element (a,a) of $D\times D$.

(10) is not symmetric as you can see that (10) contains the element (c,d) but it does not contain its symmetric partner (d,c).

(10) is antisymmetric, notice that there are no pairs of symmetric partners in (10).

(10) is not transitive. You can see that (10) contains (a,c) and (c,d) but it does not contain (a,d).

$(11) = \{(a,a),(b,b),(c,c),(c,d)\}$ is antisymmetric and transitive, but not reflexive or symmetric.

(11) is not reflexive because it does not contain the element (d,d) of $D \times D$.

(11) is not symmetric as you can see that (11) contains the element (c,d) but it does not contain its symmetric partner (d,c).

(11) is antisymmetric as you can notice that there are no pairs of symmetric partners in (11). Although (a,a), (b,b) and (c,c) are in (7), you can remember from the study guide that these pairs are acceptable in antisymmetric relations.

(11) is transitive. You can see this as (11) contains (c,c), (c,d) it also contains (c,d). There are no other pairs like this to check in (11) and so it is transitive.

The universal relation $(12) = D \times D$ is reflexive, symmetric and transitive but not antisymmetric.

(12) is reflexive as you can see that it contains (a,a), (b,b), (c,c) and (d,d) (as it contains every possible pair); and so (12) contains (x,x) for all *x* in the set *D*.

(12) is symmetric. You can see this because as (12) contains every possible pair (x,y) where *x* and *y* are in the set *D*, it must also contain its symmetric partner (y,x).

(12) is not antisymmetric as you can see that it contains the symmetric partners (a,b) and (b,a), where *a* and *b* are different.

(12) is transitive. You can see that for any (x,y) and (y,z) in (12), (x,z) must also be in (12) as it contains every possible pair from $D \times D$.

3. $(13) = \{(a,b) \in \mathbb{R} \times \mathbb{R} : a - b = 0\}$ is reflexive, antisymmetric, symmetric and transitive. It is both an equivalence relation and a partial order.

(13) is reflexive as you can see that $a - a = 0$ for all real numbers *a* in the set $\mathbb{R}$. So (13) must contain (a,a) for all *x* in the set $\mathbb{R}$.

(13) is symmetric. You can see this because if $a - b = 0$, then it is true $b - a = 0$ as well. So if (a,b) is in (13) then its symmetric partner (b,a) must also be in (13).

(13) is antisymmetric as you can see that if $a - b = 0$, then it is true that $a = b$. So every element in (13) is of the form (a,a), where *a* is in the set $\mathbb{R}$. You can remember from the study guide that these pairs are allowed in antisymmetric relations. As (13) has no other pairs apart from those in the form (a,a), it is antisymmetric.

(13) is transitive. You can see this as if $a-b=0$ and $b-c=0$, then $a-c=0$. So if (a,b) and (b,c) are in (13), then (a,c) is also in (13).

This relation is known as the **diagonal relation** and it is both an equivalence relation and a partial order. This represents equality of two real numbers.

$(14)=\{(a,b)\in \mathbb{N}\times\mathbb{N} : a \text{ divides } b\}$ is reflexive, antisymmetric and transitive, but it is not symmetric. It is a partial order but not an equivalence relation.

(14) is reflexive as you can see that *a* divides *a* for all counting numbers *a* in the set $\mathbb{N}$. So (14) must contain (a,a) for all *a* in the set $\mathbb{N}$.

(14) is not symmetric. You can see this because 2 divides into 4 and so $(2,4)$ is in (14), but 4 does not divide into 2 and so $(4,2)$ is not in (14). So (14) is not symmetric.

(14) is antisymmetric as you can see that if *a* divides into *b*, and *b* divides into *a*, then *a* must be equal to *b*. So if (a,b) is in (14), then (b,a) is in (14) only if $a=b$. You can see that there are no symmetric partners where *a* and *b* are different and so (14) is antisymmetric.

(14) is transitive. You can see this as if *a* divides into *b* and *b* divides into *c*, then *a* must divide into *c* as well. So if (a,b) and (b,c) are in (14), then (a,c) is also in (14).

(14) is a partial order as it is reflexive, antisymmetric and transitive. As (14) is not symmetric, it is not an equivalence relation.

$(15)=\{(a,b)\in \mathbb{Z}\times\mathbb{Z}: ab\neq 0\}$ is symmetric and transitive, but it is not reflexive or antisymmetric. It is neither a partial order nor an equivalence relation.

(15) is not reflexive as you can see that $0\times 0=0$ and so the pair $(0,0)$ is not in (15) As (15) does not contain (a,a) for all whole numbers *a* in the set $\mathbb{Z}$, you can say that it is not reflexive.

(15) is symmetric. You can see this because if $a\times b\neq 0$ then $b\times a\neq 0$. So if (a,b) is in (15), then (b,a) is also in (15).

(15) is not antisymmetric as you can see that both $(1,5)$ and $(5,1)$ are in (15). As these are symmetric partners where the two numbers are different, you can say that (15) is not antisymmetric.

(15) is transitive. You can see this as if $a\times b\neq 0$ and $b\times c\neq 0$, then none of *a*, *b*, or *c* are equal to zero and so $a\times c\neq 0$ as well. So if (a,b) and (b,c) are in (15), then (a,c) is also in (15).

(15) is neither a partial order nor an equivalence relation as it is not reflexive.

$(16) = \{(A,B) \in P(S) \times P(S) : A \subseteq B\}$ is reflexive, antisymmetric and transitive, but it is not symmetric. It is a partial order but not an equivalence relation.

(16) is reflexive as you can see that $A \subseteq A$ for all sets A in the set $P(S)$. So (16) must contain (A,A) for all A in the set $P(S)$.
(16) is not symmetric. You can see this because $\varnothing \subseteq S$ and so $(\varnothing, S)$ is in (14), but $S \not\subset \varnothing$ and so $(S, \varnothing)$ is not in (16). So (16) is not symmetric.
(16) is antisymmetric as you can see that if $A \subseteq B$, and $B \subseteq A$, then A must be the same set as B. So if (A,B) is in (16), then (B,A) is in (16) only if $A = B$. You can see that there are no symmetric partners where a and b are different and so (16) is antisymmetric.
(16) is transitive. You can see this as if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$ as well. So if (A,B) and (B,C) are in (16), then (A,C) is also in (16).

(16) is a partial order as it is reflexive, antisymmetric and transitive. As (16) is not symmetric, it is not an equivalence relation.

$(17) = \{(A,B) \in P(S) \times P(S) : |A| = |B|\}$ is reflexive, symmetric and transitive, but it is not antisymmetric (in general). It is a equivalence relation but not a partial order.

(17) is reflexive as you can see that $|A| = |A|$ for all sets A in the set $P(S)$. So (17) must contain (A,A) for all A in the set $P(S)$.

(17) is symmetric. You can see this because if $|A| = |B|$, then $|B| = |A|$. So if (17) contains (A,B) then it also contains (B,A) and so (17) is symmetric.

(17) is not antisymmetric in general as you can see that if $|A| = |B|$, and $|B| = |A|$, then A has the same number of elements of B but they may not be equal! Remember that a set is defined by its elements. So in general, (17) is not antisymmetric; it is only antisymmetric when S is the empty set or $|S| = |1|$.

(17) is transitive. You can see this as if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$ as well. So if (A,B) and (B,C) are in (17), then (A,C) is also in (17).

(17) is an equivalence relation for all sets S as it is reflexive, symmetric and transitive. It is also a partial order if S is the empty set or $|S| = 1$. As (17) is not antisymmetric for all other sets, it is not a partial order.

$(18) = \{((a,b),(c,d)) \in (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N}) : a \le b \text{ and } c \le d\}$ is reflexive, antisymmetric and transitive, but it is not symmetric. It is a partial order but not an equivalence relation.

(18) is reflexive as you can see that $a \le a$ and $b \le b$ for all counting numbers *a* and *b* in the set $\mathbb{N}$. So (18) must contain $((a,b),(a,b))$ for all (a,b) in the set $\mathbb{N} \times \mathbb{N}$.

(18) is not symmetric. You can see this because $2 \le 4$ and $3 \le 5$ so $((2,3),(4,5))$ is in (18), but $4 \ge 2$ and $5 \ge 3$ and so $((4,5),(2,3))$ is not in (18). So (18) is not symmetric.

(18) is antisymmetric as you can see that if $((a,b),(c,d))$ is in (18), then $((c,d),(a,b))$ is in (18) only if $a = c$ and $b = d$. You can see that there are no symmetric partners where *a* and *b* are different and so (18) is antisymmetric.

(18) is transitive. You can see this as if $((a,b),(c,d))$ is in (18) and $((c,d),(e,f))$ is also in (18), then you can notice that $a \le c$, $b \le d$ and also that $c \le e$ and $d \le f$. So $a \le e$ and $b \le f$ and you can say that $((a,b),(e,f))$ is also in (18).

(18) is a partial order as it is reflexive, antisymmetric and transitive. As (18) is not symmetric, it is not an equivalence relation. This relation is known as the **product order**.

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