Learning Enhancement Team

University of East Anglia

STUDENT SUPPORT SERVICE

Bridging Between Algebra and Calculus

The Laws of Logarithms

This guide describes the three laws of logarithms, gives examples of how to use them and introduces a common application in which they are used to change an exponential curve into a straight line.

Introduction

Logarithms are important tools in mathematics. They originally arose out of a need to simplify multiplication and division to the level of addition and subtraction. In the past, tables of logarithms were used to convert large numbers into logarithms. Then, by using the laws of logarithms detailed in this guide, multiplication was simplified to addition and exponentiation (raising to a power) was simplified to multiplication. These days calculators and computers can perform any arithmetical operation much faster than any human and a working use of logarithm tables is no longer needed. However, logarithms are still an essential subject in algebra. For example they are used to solve exponential equations, convert curves to straight lines and, in calculus, the logarithmic function plays a fundamental role.

In order to make the most of this guide you should be familiar with the language which is used when describing logarithms. You should also be familiar with the logarithmic transformation and how this is used to solve exponential equations. If you are unsure about these topics then the study guide: *Basics of Logarithms* can help you.

In this guide you will learn how to use the three laws of logarithms:

Law 1	$\log(AB) = \log(A) + \log(B)$
Law 2	$\log\left(\frac{A}{B}\right) = \log(A) - \log(B)$
Law 3	$\log(A^n) = n\log(A)$

In these laws specific bases have not been defined but the base you choose has to be the same in the mathematics on the left and right of the equals sign.

The first law of logarithms

The first law of logarithms relates multiplication to addition and states that the logarithm of a product of two terms A and B is the sum of the logarithms of those terms so that:

$$\log(AB) = \log(A) + \log(B)$$

This law is true for any base of logarithm so long as you use the same base throughout a calculation. You could re-write law 1 in base *a* as:

$$\log_a(AB) = \log_a(A) + \log_a(B)$$

You should remember that, because of the equals sign, this law also works from right-to-left and so can be used to combine logarithms of the same base that are added together. When quoting this law you should try to say it in full as it is often **misremembered** as "the log of *A* plus *B*". If you make the effort to quote it in full then this (very common) error can be avoided:

> The logarithm of *A* **multiplied** by *B* is the logarithm of *A* **plus** the logarithm of *B*

There is no law of logarithms which relates to $\log(A+B)$.

Example: Rewrite $\log(15)$ in terms of $\log(3)$ and $\log(5)$.

As $15 = 3 \times 5$ you can use law 1 with $A = 3$ and $B = 5$. So, for any base:

$$\log(15) = \log(3 \times 5) = \log(3) + \log(5)$$

Example: Express $\ln(x) + \ln(3)$ as a single logarithm.

As $\ln(x) + \ln(3)$ is the sum of two logarithms both in base *e* (natural logarithm), you can use law 1 to show that:

$$\ln(x) + \ln(3) = \ln(3 \times x) = \ln(3x)$$

where the answer also has to be a natural logarithm.

The second law of logarithms

In the same way that law 1 relates multiplication to addition, law 2 relates division to subtraction. The law states that the logarithm of a quotient of two terms A and B is the logarithm of the numerator minus the logarithm of the denominator so that:

$$\log\left(\frac{A}{B}\right) = \log(A) - \log(B)$$

Again, this law is true for a logarithm of any base *a*:

$$\log_a\left(\frac{A}{B}\right) = \log_a(A) - \log_a(B)$$

Similar to law 1, the equals sign means that this law also works from right-to-left. When quoting this law you should try to say it in full as it is often misremembered as "the log of *A* minus *B*". If you make the effort to quote it in full then this (very common) error can be avoided:

The logarithm of *A* **divided** by *B* is the logarithm of *A* **minus** the logarithm of *B*

There is no law of logarithms which relates to $\log(A - B)$ or $\frac{\log A}{\log B}$.

Example: Express $\log\left(\frac{x}{4}\right)$ in terms of $\log(x)$ and $\log(4)$.

This is a logarithm of a fraction with $A = x$ and $B = 4$ so you can use law 2 to show that, for any base:

$$\log\left(\frac{x}{4}\right) = \log(x) - \log(4)$$

Example: Simplify $\log_3(18) - \log_3(2)$.

As $\log_3(2)$ is subtracted from $\log_3(18)$ you can use law 2 with $A = 18$ and $B = 2$:

$$\begin{aligned}\log_3(18) - \log_3(2) &= \log_3\left(\frac{18}{2}\right) \\ &= \log_3 9 \\ &= 2\end{aligned}$$

The next section, or the logarithmic transformation, will explain why $\log_3 9 = 2$.

The third law of logarithms

The third law of logarithms relates exponentiation (raising to a power) to multiplication and states that the logarithm of *A* raised to the power of *n* is *n* multiplied by the logarithm of *A*:

$$\log(A^n) = n\log(A)$$

As with the other two laws, you need to use the same base throughout a calculation:

$$\log_a\left(A^n\right) = n\log_a(A)$$

This law also works from right-to-left and can be used to change a number that is multiplied by a logarithm into exponentiation; law 3 is often thought of as the most important because of this. It is especially useful when changing relationships described by curves into straight lines. The final section of this guide shows you how to change an exponential function into a straight line.

Example: Calculate $\log_3(9)$.

This may not seem like a question where you can use law 3. However, as $9 = 3^2$ you can use law 3 with $A = 3$ and $n = 2$ to find that:

$$\log_3(9) = \log_3\left(3^2\right) = 2\log_3(3)$$

and as $\log_3 3 = 1$ (see study guide: *Basics of Logarithms*):

$$\log_3(9) = 2 \times 1 = 2$$

This example can also be used to show that law 3 is an extension of law 1. You know that 9 can be written as 3×3 and so:

$$\log_3(9) = \log_3(3 \times 3)$$

Now, applying law 1 gives:

$$\log_3(3 \times 3) = \log_3(3) + \log_3(3)$$

Which again is $2\log_3(3)$.

Example: Express $\log(x) + 2\log(y)$ as a single logarithm.

There is no rule that allows you to directly combine these two logarithms because of the coefficient of 2 in the second term. If you are unsure, look back at the precise form of law 1. However the second term $2\log(y)$ is in the form $n\log(A)$ and so you can use law 3 with $A = y$ and $n = 2$ to bring the coefficient “inside” the logarithm:

$$\log(x) + 2\log(y) = \log(x) + \log\left(y^2\right)$$

Now this *is* two logarithms added together and so you *can* use law 1 with $A = x$ and $B = y^2$:

$$\log(x) + \log\left(y^2\right) = \log\left(xy^2\right)$$

Example Evaluate $2\log(4)+\log(8)+\log(50)$

You cannot combine the logarithms with law 1 because of the multiple of 2 in the first term. However, you can use the third law (as in the previous example) to show that:

$$\begin{aligned}2\log(4)+\log(8)+\log(50) &= \log\left(4^2\right)+\log(8)+\log(50)\\ &= \log(16)+\log(8)+\log(50)\\ &= \log(16\times 8\times 50)\\ &= \log(6400)\end{aligned}$$

Where the third step uses law 1 twice.

Changing exponential equations to straight lines

A very common use of logarithms in science and other disciplines is using them to transform curves into straight lines. This is particularly useful if you are studying exponential growth or decay. The equation $y = Ae^{kx}$ describes exponential behaviour and is extremely common in biology, chemistry, physics and economics. Usually A and k are constants (numbers) which describe properties of your system: A is the initial condition of the system (for example initial population) and k is a factor which affects the curve (positive k gives **exponential growth** and negative k is **exponential decay**).

The advantage of expressing a curve, such as an exponential function, as a straight line is that you can determine the gradient and intercept of a straight line easily (see study guides: *What is a Straight Line?* and *Finding Equations of Straight Lines*). These properties are connected to A and k which are difficult to determine from the curve itself. The following example gives the mathematics involved in using logarithms to change exponential graphs into straight lines and also helps you to see the correct axes to use in your plot.

Example: Use logarithms to re-express $y = Ae^{kx}$ as a straight line graph.

In order to be a straight line graph a mathematical equation has to fit the pattern $Y = mX + c$ where m is the gradient of the straight line and c is its intercept on the Y-axis. The capital letters X and Y are used to distinguish the variables from the y and x used in the exponential equation. To make $y = Ae^{kx}$ fit this pattern, you first need to take logarithms (you would say "taking logs") of each side of the equation, this involves writing ln (as you have a base of e it is wise to use natural logarithms) in front of either side of the equation:

$$\ln(y) = \ln(Ae^{kx})$$

$\ln(y) = \ln(A) + \ln(e^{kx})$ using Law 1 to make the right-hand side into addition

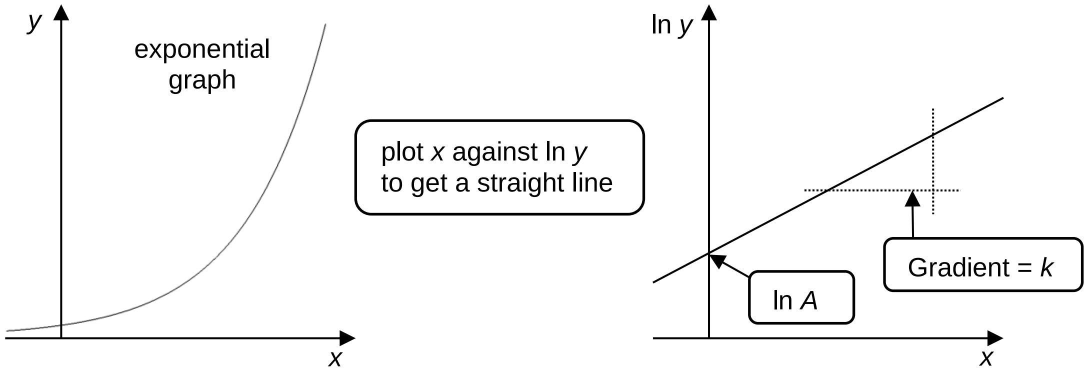
$\ln(y) = \ln(A) + kx\ln(e)$ using Law 3 on the second term of the right-hand side

$\ln(y) = \ln(A) + kx$ since $\ln e = 1$

Although this may not be immediately obvious, your equation now has the form of a straight line. You can see this by writing the two together, one above the other:

$$\ln y = kx + \ln A$$
$$Y = mX + c$$

Here the pieces of the equation line up in the same pattern where $Y = \ln y$, $m = k$, $X = x$ and $c = \ln A$. So if you plot a graph with $\ln y$ on the vertical (Y) axis and x on the horizontal (X) axis you get a straight line with a gradient of k and a vertical intercept of $\ln A$. So the constants A and k, which are difficult to calculate directly from the curve, can be found easily from the straight line.



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