University of East Anglia
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Bridging Between Algebra and Calculus

Basics of Logarithms

This guide describes logarithms and their basic properties. It identifies the link between logarithms and exponential functions. It shows how to solve exponential equations using logarithms.

Introduction

Before the invention of the calculator, methods for shortening the processes of multiplication and division were important in mathematics. In 1593 two Danish mathematicians used trigonometric tables to reduce multiplication and division to the level of the easier skills of addition and subtraction. This method inspired the mathematicians John Napier and Jobst Bürgi to independently develop tables based on indices which performed a similar purpose. Napier named these numbers **logarithms**.

This study guide will help you understand the link between logarithms and exponents. It will also show you how to switch between the logarithmic and exponential forms of a particular piece of mathematics. This study guide assumes a good knowledge of the properties of exponents, the study guide: *Laws of Indices* can help you with this. Logarithms exploit the laws of indices to transform multiplication into addition and division into subtraction.

Logarithms and exponents are closely linked. Just as subtraction is the inverse operation of addition and division is the inverse operation of multiplication, logarithms are the inverse operation of exponentiation, which is raising a number to a power. Because of this you can use logarithms to help you solve **exponential equations** (see later in this guide).

Exponential equations

One the most important uses of logarithms are as a tool to solve exponential equations. An exponential equation is one where the unknown is part of an exponent within the equation. For example:

$$10^x = 100 \qquad 4^{2x} = 9 \qquad 6^{1-x} = 6.2$$

are all exponential equations as the unknown x is part of the exponent. Exponential equations can be very easy or very difficult to solve and there tends to be no in between. A slight change in an equation which is easy can make it very difficult to solve.

Example: Solve $10^x = 100$.

This is an example of an exponential equation as the unknown *x* is part of the exponent. Remember that $10^2 = 100$ and so the solution is $x = 2$. This type of exponential equation is easy to solve. (However it is common to give the answer as $x = 10$ as $10 \times 10 = 100$; it is important to remember that you are dealing with indices here, not multiplication.)

A very minor change to this example can make its solution very difficult.

Example: Solve $10^x = 101$.

Solving this by hand is very difficult, but you can use logarithms to help.

Using logarithms to solve simple exponential equations

You can write a general exponential equation as:

$$y = a^x$$

where, in the previous example $y = 101$, $a = 10$ (called the **base** of the equation) and the exponent is $x = x$. In mathematics there is an equivalent way of writing this general exponential equation in terms of a logarithm. This is:

$$x = \log_a y$$

here the exponent in the first equation, *x*, is equal to a logarithm which is written as "log", the subscript indicates that the logarithm has a **base of *a*** and you are taking the **logarithm of *y***. There is no multiplication here as taking a logarithm is a different operation in mathematics. You would pronounce the notation $\log_a y$ as "log to the base *a* of *y*". The base of a logarithm can be any positive number, never negative. You can move freely between the two types of equations using the **logarithmic transformation**:

$$y = a^x \Leftrightarrow \log_a y = x$$

If you have an exponential equation, expressing it in logarithmic form has an advantage as $\log_a y$ can usually be found using a calculator.

Before you attempt to solve $10^x = 101$, you should make sure that you understand how to switch between the exponential and logarithmic forms of equations.

Example: Convert the exponential equation $10^x = 100$ to logarithmic form.

In the exponential equation $10^x = 100$ the base $a = 10$, the exponent $x = x$ and $y = 100$.

So you can use the logarithmic transformation:

$$100 = 10^x \Leftrightarrow \log_{10} 100 = x$$

You know from the first example in this guide that $x = 2$ and so you can write that $\log_{10} 100 = 2$. You can think of the mathematics $\log_{10} 100$ asking the question "what power do you have to raise 10 to to get 100" and the answer is 2. Similarly the expression $\log_7 49$ Asks the question "what power do you have to raise 7 to to get 49", the answer is also 2.

Example: Solve $10^x = 101$.

Now let's return to the example from before but now you can use the logarithmic transformation with $a = 10$, $x = x$ and $y = 101$:

$$101 = 10^x \Leftrightarrow \log_{10} 101 = x$$

You should notice that the left hand side of $\log_{10} 101 = x$ is just a number. This number is not obvious though as it is the power that you need to raise 10 to to get 101. You can, and should, use a calculator to find it. Calculators calculate logarithms in various ways and you should consult the instruction manual of your calculator to help you work out how to do it. You should get $x = 2.004$ to 3 decimal places. To check the answer, use your calculator again to find that $10^{2.004} \approx 101$ (the answer is only approximate as you have rounded the 2.004).

Example: Solve $\frac{1}{25} = 6^x$.

Even though this example involves working with fractions the logarithmic transformation works in exactly the same way. Here the base $a = 6$, the exponent $x = x$ and $y = 1/25$ so:

$$\tfrac{1}{25} = 6^x \Leftrightarrow \log_6 \tfrac{1}{25} = x$$

And you can use your calculator to see that $x = \log_6 \frac{1}{25} = -1.796$ to 3 decimal places.

Solving more complicated equations with logarithms

Many exponential equations contain complicated exponents. You can solve these equations using the logarithmic transformation in exactly the same way as in the previous section.

Example: Solve $8^{3x-2} = 24$.

Here it is not easy to find what *x* has to be to satisfy the equation. However expressing the exponential equation as its equivalent logarithmic form is useful. So using a base of 8 and an exponent of $3x - 2$ the logarithmic transformation gives:

$$8^{3x-2} = 24 \Leftrightarrow \log_8 24 = 3x - 2$$

You can solve $\log_8 24 = 3x - 2$ for x by conventional rearranging (see study guide: *Rearranging Equations*). (You need to subtract two from each side and then divide each side by 3.) This gives:

$$x = \tfrac{1}{3}\left[(\log_8 24) + 2\right]$$

which may look complicated but it is just a number. Careful inputting into a calculator gives $x = 1.176$ to 3 decimal places. To check, put this result back into the original question to find that:

$$8^{(3\times 1.176)-2} \approx 24$$

You should notice the use of brackets to help you in the above calculations. Brackets are very useful when dealing with logarithms, for example $(\log_8 24) + 2$ and $\log_8(24 + 2)$ are not equivalent.

Properties of logarithmic functions

You can use specific values of *a* and *x*, along with their connection with exponents, to find special properties of the logarithmic function.

Property 1: **The logarithm of 1 is zero, regardless of the base**

From the laws of indices you know that $a^0 = 1$, in other words raising any number to the power of 0 gives 1. For example $4^0 = 1$, $100^0 = 1$ and so on. You can use this in the logarithmic transformation to give:

$$a^0 = 1 \Leftrightarrow \log_a 1 = 0$$

which reveals that the logarithmic equivalent of the exponential equation $a^0 = 1$ is $\log_a 1 = 0$. In other words, regardless of the base, the logarithm of 1 is zero.

Example: What is $\log_{14} 1$?

As the logarithm of 1 is zero regardless of base, $\log_{14} 1 = 0$.

Property 2: **The logarithm to base *a* of *a* is 1**

From the laws of indices you know that $a^1 = a$, in other words raising any number to the power of 1 gives that number. For example $4^1 = 4$, $100^1 = 100$ and so on. You can use this in the logarithmic transformation to give:

$$a^1 = a \Leftrightarrow \log_a a = 1$$

which reveals that the logarithmic equivalent of the exponential equation $a^1 = a$ is $\log_a a = 1$. In other words, the logarithm to base *a* of *a* is 1.

Example: What is $\log_8 8$?

Using $\log_a a = 1$ with $a = 8$ gives $\log_8 8 = 1$.

Important bases for logarithms

The two most common bases for logarithms are 10 and *e* (where *e* is Napier's constant, $e = 2.71828...$,). Firstly, logarithms with a base 10 are called **common logarithms** and are commonly used to manipulate scales which go from the very small to the very large. On your calculator the common logarithm is usually denoted by **log** button.

Logarithms with a base of *e* are called **natural logarithms**. The number *e* is one of the most important numbers in mathematics and is related to compound interest calculations, amongst many other things. Exponential equations with *e* as a base describe exponential growth (such as populations) and decay (such as the decay of radioactive isotopes) in the sciences, see study guide: *Exponential Functions*. The natural logarithm offers a way to manipulate exponential equations with a base of *e*. It is common in science to see natural logarithms used to change the curve of an exponential equation to a straight line, which helps in an analysis. Natural logarithms also play a crucial role in mathematics as they are the only logarithms which evolve out of calculus. On your calculator the natural logarithm is usually accessed via the **ln** button.

Changing the base of a logarithm

The numbers 10 and *e* are not the only bases for logarithms. In fact any positive number can serve as a base for a logarithmic function. Most modern calculators can calculate logarithms of any base however some do not. You should not worry as logarithms are flexible and you can change a logarithm from any base to either base 10 or base *e*, you can then use your calculator to perform the calculation in the accessible base. In general if you want to change the base of a logarithm from *a* to *b* you use:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

The logarithm of a number *x* for base *a* is equivalent to the logarithm of the same number *x* in base *b* divided by the logarithm of its original base *a* in base *b*. Now, by replacing the base *b* with the common logarithm base 10 or the natural logarithm base *e*, you can use your calculator to obtain the value of $\log_a x$.

To change from base *a* to base 10

$$\log_a x = \frac{\log_{10} x}{\log_{10} a}$$

To change from base *a* to base *e*

$$\log_a x = \frac{\ln x}{\ln a}$$

Importantly, you can use either of these equations and you will always get the same answer.

Example: Find the value of $\log_3 7$

Firstly let's change to base 10:

$$\log_3 7 = \frac{\log_{10} 7}{\log_{10} 3} = \frac{0.845...}{0.477...} = 1.771 \text{ to 3 decimal places}$$

Now let's use natural logarithms:

$$\log_3 7 = \frac{\ln 7}{\ln 3} = \frac{1.946...}{1.099...} = 1.771 \text{ to 3 decimal places}$$

Check the answer by finding that $3^{1.771} \approx 7$.

Want to know more?

If you have any further questions about this topic you can make an appointment to see a **Learning Enhancement Tutor** in the **Student Support Service**, as well as speaking to your lecturer or adviser.

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