

Bridging Between Algebra and Calculus

Exponential Functions

This guide explores the basic properties of exponential functions and how to use them in calculations using examples from biology and economics. It also explains the connection between exponential functions and logarithmic functions.

Introduction

Exponential functions are an extremely important family of functions in both pure mathematics and where mathematics is applied. They are similar to the power function $y = ax^n$, however, in an exponential function, the unknown variable x is part of the index of the function, not the base. You can write a **general exponential function** as:

$$y = Ab^{kx}$$

Where b is a number called **the base** and the variable x forms part of **the index** or **exponent** of the function. This formula also contains two constants and it is important that you understand their meaning properly. The first constant A is the value of the exponential function when $x = 0$ and is where the function crosses the y -axis. The other constant is k and this controls aspects of the shape of the function such as its steepness and importantly whether the exponential function describes **exponential growth** or **exponential decay**.

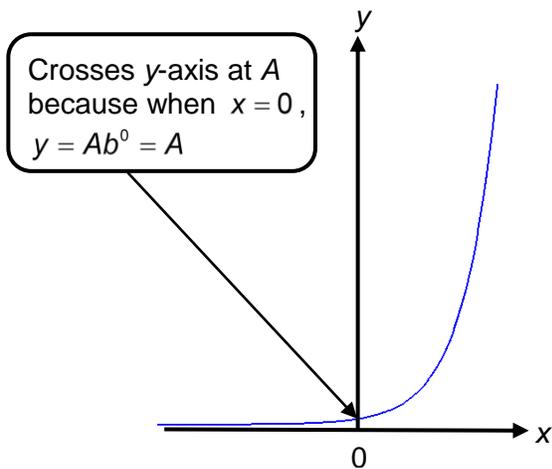
For exponential growth the value of k is positive, $k > 0$.

For exponential decay the value of k is negative, $k < 0$.

Exponential behaviour is common in applied mathematics, for example it is seen in:

- compound interest in economics.
- reaction kinetics in chemistry.
- population dynamics or spread of infection in biology.
- nuclear chain reactions and radioactive decay in physics.

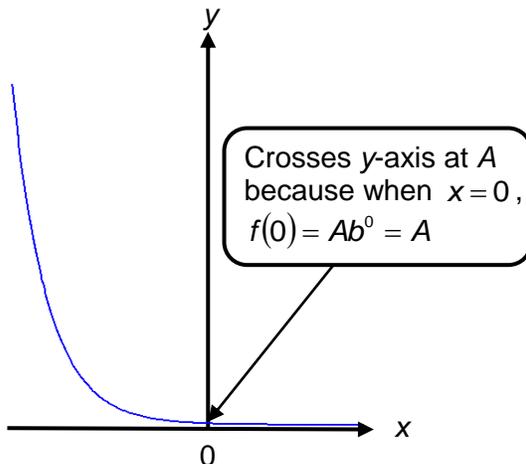
The simplest example of exponential growth can be seen in the graph of $y = Ab^x$, shown below, which is obtained by setting $k = 1$ in the general exponential function:



As x increases, the value of the exponential function increases rapidly.

The x -axis is an **asymptote** (see study guide: [Sketching a Graph](#)). This is because as x gets more negative, the value of y gets closer to 0. However there is no value of x which gives the function a value of 0. So the graph never crosses the x -axis. There is no asymptote as x approaches infinity.

The simplest example of exponential decay can be seen in the graph of $y = Ab^{-x}$, shown below, which is obtained by setting $k = -1$ in the general exponential function:



As x increases, the value of the exponential function rapidly approaches 0.

Again, the x -axis is an asymptote. This is because as the value of x increases, the value of y gets closer to 0. However there is no value of x which gives the function a value of 0. This graph is the graph of the function $y = Ab^x$ reflected in the y -axis. There is no asymptote as x approaches negative infinity.

Exponential functions and the number e

An important example of a family of exponential functions is when the base is equal to the special number e . This general function, sometimes called the **natural exponential function**, is very common in mathematics, economics and science and is written as:

$$y = Ae^{kx}$$

where A and k are constants as explained above.

The number e (always lower case) is the symbol for **Euler's number** (named after the famous mathematician Leonhard Euler) and has the value of:

$$e = 2.7182818285\dots$$

It is a common misconception that the e stands for “exponential” but in fact it stands for “Euler”. Euler was the first person to use this notation in the 18th century and he was able to approximate the value of e to 18 decimal places. The number e has a crucial role in mathematics, especially calculus, in a similar way that π is central to geometry and is both an irrational and transcendental number (see study guide: [Different Kinds of Numbers](#)).

Examples of exponential behaviour

Values or measurements that **grow exponentially** exhibit behaviour which increases proportionally over a constant amount of time, for example. As a direct effect of this, the observation you are measuring grows rapidly. You can see this by thinking about this simple system; start with the number 5 and double it every 1 minute, the sequence below shows the numbers you have as the minutes tick by:

5 10 20 40 80 160 320 640 ...

The numbers are said to be **growing exponentially** and you can probably see that they will get extremely big extremely quickly. The exponential function which describes this system is:

$$y = 5 \cdot 2^t$$

where the variable time is represented by t . The number you get after a certain time is y and depends on raising the base 2 to the power of t . The 2 comes from the fact that you are doubling and the value of A is 5 because you started with the number 5.

Values or measurements that **decay exponentially** lose a constant proportion of their value in a given time for example. Think about the number 4000, if you halve it every minute then you get the following sequence of numbers as time goes by:

4000 2000 1000 500 250 125 ...

the numbers are said to be **decaying exponentially**. You can see that the numbers are getting smaller but the rate at which this is happening is getting slower as the number you are halving is getting smaller each time. More subtly, the numbers in the sequence will never reach zero as you will always have something left over after you half the previous number. The exponential function which describes this system is:

$$y = 4000 \cdot 2^{-t}$$

Where again the variable time is represented by t . The number you get after a certain time is y and depends on raising the base 2 to the power of $-t$. The 2 comes from the fact that you are halving and the value of A is 4000 because you started with the number 4000.

Exponential functions and logarithms

If you divide both side of the exponential function by A you get:

$$\frac{y}{A} = b^{kx}$$

This is useful because in mathematics there is an equivalent way of writing an equation of this kind in terms of a single **logarithm**. This is:

$$kx = \log_b \left(\frac{y}{A} \right)$$

This piece of mathematics is discussed further in the study guide: [Basics of Logarithms](#). You can move freely between the two types of equations using the **logarithmic transformation**:

$$\frac{y}{A} = b^{kx} \Leftrightarrow \log_b \left(\frac{y}{A} \right) = kx$$

Importantly exponential functions and logarithms are mathematical inverses of each other (see study guide: [Inverse Functions and Graphs](#)). In other words they undo each other in the same way that addition undoes subtraction. This is extremely useful when you are rearranging or solving exponential equations.

Example: In the exponential growth example in the previous section, how many minutes does it take for the numbers to exceed 5000?

The growth of the numbers is given by $y = 5 \cdot 2^t$. So with $y = 5000$ and $A = 5$ you can write, after dividing both sides by 5:

$$1000 = 2^t \Leftrightarrow \log_2 1000 = t$$

Inputting $\log_2 1000$ into a calculator gives $t = 9.97$ to 2 decimal places, so it takes 9.97 minutes for the numbers to exceed 5000.

Like all other exponential functions, the natural exponential function is linked to a logarithm via the logarithmic transformation. In mathematics this logarithm is called the

natural logarithm. The natural logarithm has base e and is written using the symbol “ln” rather than “ \log_e ”. So, using the logarithmic transformation:

$$y = e^x \Leftrightarrow \ln y = x$$

Exponentials in action: Populations

It is very common for exponential equations to describe how population size changes over time. Often t (for time) replaces x as one variable and the population P_t (with the subscript indicating the population at a time t) replaces the other variable y . In this case you will often see the exponential function written as:

$$P_t = P_0 b^{kt}$$

Here P_0 represents the initial population i.e. the population at time $t = 0$. Note that P_0 is equivalent to the constant A introduced in the general exponential function.

Example: The population of bacteria in a sample increases by 10 times every hour. How many bacteria are present after 20 hours if you begin with a culture of 2 bacteria?

In this example, you know that you have 2 bacteria to begin with, when $t = 0$, and so $P_0 = 2$. You also know that the population increases by a factor of 10 every hour which tells you that 10 is the base. So the exponential equation for this system is:

$$P_t = 2 \cdot 10^{kt}$$

You need to find the population when $t = 20$ which is P_{20} . The problem here is that you do not know the value of k . However you do know that after one hour ($t = 1$) there are 20 bacteria, in other words, $P_1 = 20$. You can use this to find k :

$$P_1 = 20 = 2 \cdot 10^k$$

Which means that, after dividing by 2, $10 = 10^k$ implying that $k = 1$. So, using this and $t = 20$ you can calculate the number of bacteria after 20 hours:

$$P_{20} = 2 \cdot 10^{1 \cdot 20} = 2 \cdot 10^{20} = 200\,000\,000\,000\,000\,000\,000$$

A lot of bacteria!

Exponentials in action: The compound interest formula

Exponential growth is an important concept in economics where it is applied to

continuously compounded interest and to inflation. When banks advertise interest rates it is often expressed as $r\%$ per annum. So that if, for example, £100 was invested for a year, at the end of the year it would be worth £105 if you got 5% interest paid once only. If interest is paid **continuously**, then the money grows **exponentially** according to the formula:

$$P = P_0 e^{rt}$$

where P is the amount of money the investment is worth at time t measured in years, for an interest rate r written as a decimal and for an initial investment P_0 .

Example: You take out a £100 loan from the bank compounded continuously with interest at 5%. How much will you have to pay back after 6 months?

In this example, interest is compounded continuously for six months and so $t = 0.5$ years at an interest rate of $r = 5\% = 0.05$ and you borrow $P_0 = £100$. So, using the formula above:

$$P = £100e^{0.05 \times 0.5} = £102.53$$

Which means that you pay back £102.53 after 6 months.

Want to know more?

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