Learning Enhancement Team

Bridging Between Algebra and Calculus

Simultaneous Equations

This guide introduces simultaneous equations and describes a general method to help to solve simple, linear simultaneous equations.

Introduction

An **equation**, a piece of mathematics with an equals sign in it, describes a graph. Different types of equations describe different types of graphs. For example all **straight lines** have the general equation $y = mx + c$ (where m is the gradient of the line and c is the y-intercept) and all **parabolas** have the general equation $y = ax^2 + bx + c$ (see study guides: *What is a Straight Line?* and *Quadratic Functions* for more details). Other functions have distinctive graphs which you should familiarise yourself with (see factsheet: *Five Basic Functions*).

Another idea in mathematics that you should be comfortable with is the meaning of **solving** an equation or set of equations. In order to solve a single equation you need to have one equation and one unknown for example you can solve the equation $5x - 3 = 0$ as there is only one unknown x but you cannot solve an equation like $y = 5x - 3$ because there are two unknowns, x and y.

If you want to solve equations with two unknowns you need two equations. So you can solve a pair of equations such as:

$$y = 5x - 3 \qquad \text{and} \qquad y = 2x + 4$$

These two equations are both in the form $y = mx + c$, so their graphs are straight lines, and are known as **simultaneous equations**. The solution to these types of simultaneous equations is usually a single value of x and a single value of y which represents a point which occurs on each graph. If a point is on both lines then the two lines must **intersect** at that point.

Solutions to simultaneous equations are the points of intersection of lines the equations represent.

So the mathematics of simultaneous equations is the same as that of lines. In fact the subject of solving simultaneous equations is called **Linear Algebra –** the algebra of lines.

In this guide you will learn:

1) **How to find the point of intersection of two straight lines** by solving simultaneous equations which are in the form $y = mx + c$.
2) **How to tell if two lines are parallel.** As parallel lines do not intersect, they give rise to simultaneous equations which have no solutions.

Different types of simultaneous equations

When you are trying to solve linear simultaneous equations it is extremely helpful to express them in the form $y = mx + c$. You may need to transpose your equations to make *y* the subject (see study guide: *Rearranging Equations* for help with this). The reason for doing this is that is you can identify the gradient and *y*-intercept of each graph easily and these values can help you to find the solution. It is also useful if you can sketch the lines to get an idea of what the question looks like graphically, see study guide: *Sketching Straight Lines*.

(1) Equations with different gradients and different y-intercepts

Example: Examine the simultaneous equations $y - 3 = 2x$ and $y + x + 6 = 0$ and comment on their graphs.

The first step towards learning about simultaneous equations is to transpose them for *y*. So:

$y - 3 = 2x$	becomes	$y = 2x + 3$	(1)
$y + x + 6 = 0$	becomes	$y = -x - 6$	(2)

It is useful to label your equations, in this case (1) and (2), as you can easily refer back to them within your answer. Now you have them in this form you can see that *y* is equal to both $2x + 3$ and $-x - 6$. As both of these expressions are equal to *y* you can set them equal to each other to give:

$$2x + 3 = -x - 6$$

which only has one unknown *x* and so can be solved. A good method for solving an equation like this is to collect all the unknowns on one side and all the constants (numbers) on the other. So adding *x* to each side and then subtracting 3 from each side puts all the unknowns on the left and all the constants on the right:

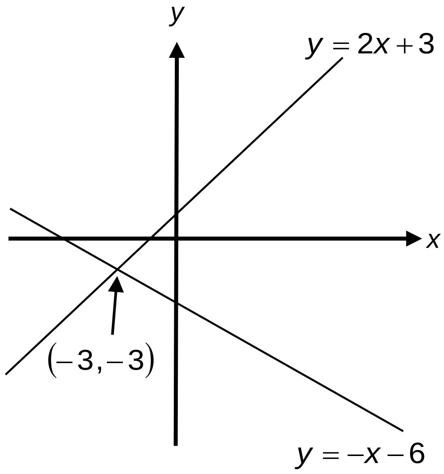
$$3x = -9$$

To finish, divide both sides by 3 to find that $x = -3$. This is the first part of the solution, the *x*-coordinate. You should realise that a solution to simultaneous equations of this type comprises two parts, a value for *x* and a value for *y*. You need to now find a value for *y*.

Since you have a value for *x* you can choose either (1) or (2) and substitute *x* into the equations to find *y*.

Choosing (1) $y = 2x + 3 = 2 \cdot (-3) + 3 = -6 + 3 = -3$
Choosing (2) $y = -x - 6 = -(-3) - 6 = 3 - 6 = -3$

It does not matter which equation you choose, each gives the value of *y* as -3. It is true that you only need to pick one equation to find the other unknown. However you can use the other equation to check your answer as it should give you the same result. The solution $y = 2x + 3$ and $y = -x - 6$ is $x = -3$ and $y = -3$. This result says that the graphs of these functions cross at the point $(-3, -3)$ as shown in the sketch on the right.



Simultaneous equations with different gradients and *y*-intercepts are the most common types students are asked to solve.

Linear simultaneous equations with different gradients and different *y*-intercepts always cross at a single point

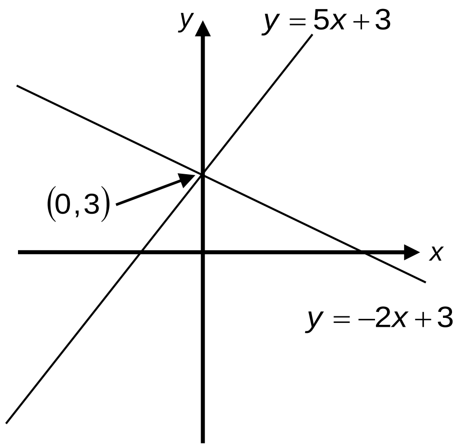
(2) Equations with different gradients and identical *y*-intercepts

Example: Examine the simultaneous equations $y + 2x = 3$ and $y - 3 = 5x$ and comment on their graphs.

Again, to gain insight into the simultaneous equations you must first transpose them and make *y* the subject:

$y + 2x = 3$	becomes	$y = -2x + 3$	(1)
$y - 3 = 5x$	becomes	$y = 5x + 3$	(2)

This shows that the graphs for (1) and (2) have different gradients (-2 and 5 respectively) but the same *y*-intercept of 3. As the *y*-intercepts are the same, the graphs must cross at this point (see picture to the right). As the solutions to simultaneous equations are points where lines intersect each other, the *y*-intercept must be a solution. Here the *y*-intercept is the coordinate $(0, 3)$ and so the solution is $x = 0$ and $y = 3$ which can be easily checked by substitution.



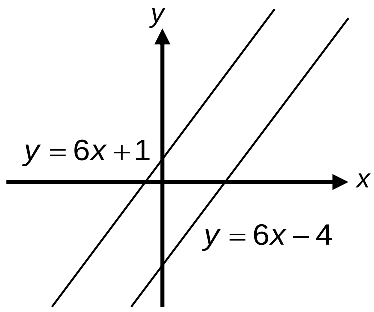
Linear simultaneous equations with different gradients and identical *y*-intercepts always and only cross at the *y*-intercept

(3) Equations with identical gradients but different *y*-intercepts

Example: Examine the simultaneous equations $y-1=6x$ and $y/2=3x-2$ and comment on their graphs.

Again, to gain insight into the simultaneous equations you must first transpose them and make *y* the subject:

$y-1=6x$	becomes	$y=6x+1$	(1)
$y/2=3x-2$	becomes	$y=6x-4$	(2)



The transposition shows that the graphs for (1) and (2) have the same gradient of 6 but different *y*-intercepts (1 and -4 respectively). Graphically these represent parallel lines (as seen to the right). As parallel lines do not intercept you would expect there to be no solutions to the simultaneous equations. If you set the two expressions for *y* equal to each other you find that $6x+1=6x-4$. After subtracting $6x$ from each side you get $1=-4$ which cannot be true. As your mathematics leads to a result which cannot be true, you cannot solve (1) and (2).

As a solution to a simultaneous equation represents where their graphs cross it cannot be true that the graphs cross and so they must be parallel.

Linear simultaneous equations with identical gradients and different *y*-intercepts represent parallel lines and do not cross

(4) Equations with identical gradients and identical *y*-intercepts

Example: Examine the simultaneous equations $3y+9=12x$ and $(y-1)/4=x-1$ and comment on their graphs.

Again, to gain insight into the simultaneous equations you must first transpose them and make *y* the subject:

$3y+9=12x$	becomes	$y=4x-3$	(1)
$(y-1)/4=x-1$	becomes	$y=4x-3$	(2)

This shows that (1) and (2) are identical and so they have the same gradient of 4 and a *y*-intercept of -3. Setting the two expressions for *y* equal to each other you get $4x-3=4x-3$, which is always true. This implies that there are an infinite number of solutions to (1) and (2) as there are an infinite number of points which occur on both lines.

Linear simultaneous equations with identical gradients and identical *y*-intercepts are the same line and have infinite solutions

Generalisation

If you have two linear equations you can always write them as:

$$y = mx + c \qquad (1)$$
$$y = nx + d \qquad (2)$$

where equation (1) has a gradient of *m* and a *y*-intercept of *c* and equation (2) has a gradient of *n* and a *y*-intercept of *d*. You need to transpose your equations to make *y* the subject before you decide the values of *m*, *n*, *c* and *d*.

As shown in the examples above, knowing *m*, *n*, *c* and *d* can help you understand the simultaneous equations as illustrated in the following table:

Gradients	***y*-intercepts**	**Type**	**Comments**
$m \neq n$	$c \neq d$	(1)	The lines cross at a single point not on the *y*-axis.
$m \neq n$	$c = d$	(2)	The lines cross at the *y*-intercept $(0,c)=(0,d)$.
$m = n$	$c \neq d$	(3)	The lines are parallel and so you get no solutions.
$m = n$	$c = d$	(4)	The lines are identical and so you get infinite solutions.

When straight lines cross, in types (1) and (2), they only cross at a single point with the coordinate (x,y). When you solve simultaneous equations you find the relevant values of *x* and *y* and hence you can find the coordinate. Given the two equations $y = mx + c$ and $y = nx + d$, when the lines cross the value of *y* is the same for both equations. You can write this another way by equating the right-hand sides of the equations as they are both equal to *y*:

$$mx + c = nx + d$$

When $m = n$ and $c = d$ you find the equation above becomes $mx + c = mx + c$ which is always true and so has infinite solutions.

When this is not the case the advantage of writing $mx + c = nx + d$ is that you now only have one variable x and so you can transpose this equation to find it. This gives a general solution for x of:

$$x = \frac{d - c}{m - n} \qquad (3)$$

Substituting the values of c, d, m and n into (3) gives the x-coordinate of the intersection point. You can also use this equation to find the value of y. By choosing either $y = mx + c$ or $y = nx + d$ and substituting in x you find that:

$$y = \frac{dm - cn}{m - n} \qquad (4)$$

You can use equations (3) and (4) to help you analyse all the different types of simultaneous equations.

When $m \neq n$ and $c = d$ equation (3) gives $x = 0$ and (4) gives $y = c = d$ which means that the lines cross at the y-intercept.

When $m = n$ and $c \neq d$ equations (3) and (4) involve dividing by zero and so cannot work indicating parallel lines.

Want to know more?

If you have any further questions about this topic you can make an appointment to see a **Learning Enhancement Tutor** in the **Student Support Service**, as well as speaking to your lecturer or adviser.

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