Learning Enhancement Team

University of East Anglia

STUDENT SUPPORT SERVICE

*Bridging Between Algebra and Calculus*

# Inverse Functions and Graphs

***This guide introduces the concept of the inverse of a function and its relationship to its graph.***

## Introduction

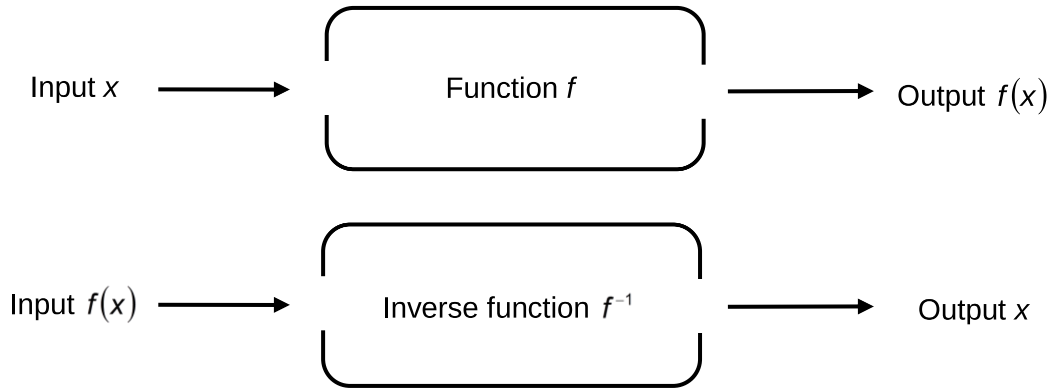
It is extremely useful in mathematics to be able to undo something that you have done. For example you can think of addition being "undone" by subtraction and multiplication being "undone" by division. In mathematics there is a specific word to describe the idea of undoing, it is **inverse**. So subtraction is the *inverse* of addition just as addition is the *inverse* of subtraction. The idea of an inverse can be extended to functions as the more complicated mathematical expressions which make a function are created from combinations of basic operations. However, do other functions such as sine, cosine, exponential and logarithm have inverses? This guide aims to show you how to determine whether a function has an inverse. Before you continue, you should make sure you are familiar with the notation and concepts associated with functions, the study guides: *Functions* and *Using Functions* can help you with this.

## The inverse of a function

If you think about a function as a machine which changes an input into a unique output then the inverse of this function would also be a machine which changes the output back into the input. Importantly, **the inverse of a function is also a function**. The idea of a function can be expressed mathematically as the function $f$ has an input $x$ and output $f(x)$. The **inverse** of this function, if it exists, is written $f^{-1}(x)$. It is very important that you realise that the superscript "$-1$" in $f^{-1}(x)$ indicates *the inverse of a function and not an index*. It is common to confuse the inverse of a function with the reciprocal function and you should always take note of the context in which the $-1$ appears: if the piece of mathematics you are looking at concerns the laws of indices in some way, the $-1$ probably represents the reciprocal, on the other hand if you are looking at functions then the $-1$ probably means inverse.

Not all functions have inverses. Only functions which are **bijections**, those which are both injections (**one-to-one**) and surjections (**onto**) have an inverse. These functions can be identified as functions where each output can be paired with **only one** input.

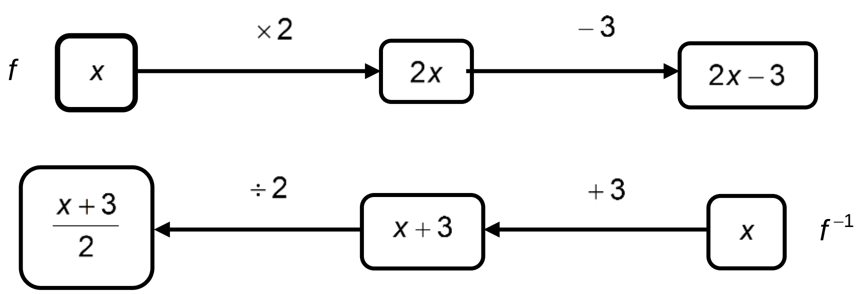
You can represent the idea of a function and its inverse pictorially:




The top diagram shows a function $f$ with an input of $x$ with the output $f(x)$. The lower diagram shows that if the output of the function $f$ acts as the input to its inverse of $f^{-1}$ then the resulting output is $x$.

*Example:* Find the inverse, if it exists, of $f(x) = 2x - 3$.

First of all you have to check if the function has an inverse, i.e. is it a bijection. In order to do this you have to check if each output is paired with only one input value. Here, for every input $x$ there is a unique output. So $f$ is a bijection and $f^{-1}$ exists. To find the formula of $f^{-1}$ you have to work out what operations would undo the mathematics involved in $f(x) = 2x - 3$. It is crucial to get the operations in the right order. A simple *Flow Chart Method* could be a useful tool for this process (see study guide: *Rearranging Equations*). You create the function $f(x) = 2x - 3$ by first multiplying the input by 2 and then subtracting 3. The inverse of this process is taking an input and adding 3 then dividing by 2. The flow chart for this procedure looks like this:




So the formula of the inverse function is $f^{-1}(x) = \dfrac{x+3}{2}$.

This process for finding an inverse function is not always straightforward. Sometimes you may encounter difficulties in showing that a function is a bijection, learning the language and definitions of functions can help you with this. Also finding the function formula $f(x)$ could prove to be a challenging process, becoming familiar with the flow-chart method can help to overcome this. It is important for you to know when a function does not have an inverse.

*Example:* Find the inverse of the function $f(x)=x^2$, when $x$ is a real number, if it exists.

Firstly, you have to check if the function is a bijection. The squares of numbers with opposite signs give the same result, for example $(-2)^2$ and $2^2$ both equal 4. This means that each output value is paired with two different input values. Specifically for $f(x)=4$, $x$ can either be $\pm 2$. Therefore $f(x)=x^2$ in not an injection, therefore not a bijection and thus does not have an inverse.

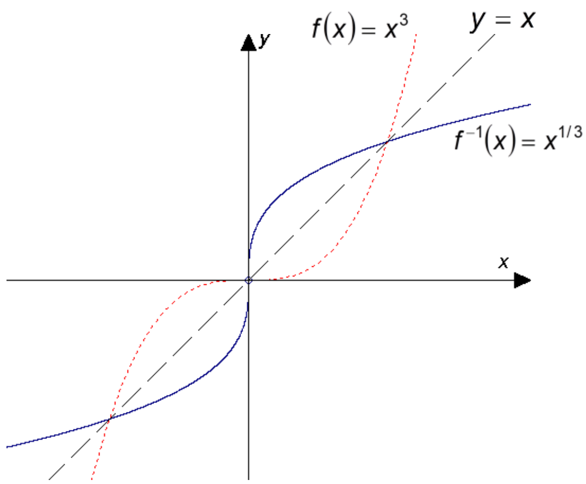
## Connection between functions, inverses and graphs

You can think of a coordinate on a graph as defined as $(x, f(x))$. As you have seen, the inverse of a function swaps the roles of input and output in the original function. As a result if the point $(x, f(x))$ belongs to the graph of a function then the point $(f(x), x)$ will belong to the graph of its inverse. There is nice way of thinking of this by considering the graphs of a function and its inverse. It is always the case that the point $(f(x), x)$ is the reflection of the point $(x, f(x))$ in the line $y=x$. A direct result of this is that the graph of $f^{-1}(x)$ is the obtained by reflecting the graph of $f(x)$ in the line $y=x$.

*Example*: Sketch the graphs of $f(x)=x^3$ and its inverse.

In the figure to the right you can see the graphs of the function $f(x)=x^3$ (dotted curve) and its inverse $f^{-1}(x)=x^{1/3}$ (solid curve).

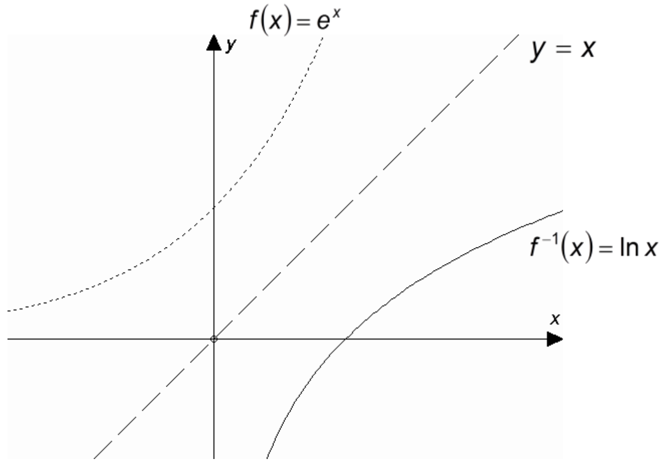
It is clear that they are reflections of each other in the diagonal $y=x$ (dashed line).




*Example:* Find the inverse of the function $f(x)=e^x$ and sketch its graph.

The function $f(x)=e^x$ is a bijection and so it has an inverse. The laws of logarithms tell you that if $f(x)=e^x$ then $\ln(f(x))=x$ and so you can write $f^{-1}(x)=\ln x$. In other words $\ln x$ undoes $e^x$ and the inverse of $f(x)=e^x$ is the function $f^{-1}(x)=\ln x$.

The graphs of these functions are shown to the right. You can see that $f(x)=e^x$ (dotted line) and $f^{-1}(x)=\ln x$ (solid line) are reflections of each other in the line $y=x$ (dashed line).




You can see that that two basic functions (the exponential and logarithmic functions) are inverses of each other. This is discussed in more detail in the study guides: *Basics of Logarithms* and *Exponential Functions*.

The trigonometric functions also have inverses which are useful to know. The inverse of $\sin x$ is $\sin^{-1} x$ (often written as $\arcsin x$). Similarly the inverse of $\cos x$ is $\cos^{-1} x$ (often written as $\arccos x$). The inverses of the trigonometric functions are discussed in more depth in the study guide: *Trigonometric Ratios: Sine, Cosine and Tangent.*

## Want to know more?

If you have any further questions about this topic you can make an appointment to see a **Learning Enhancement Tutor** in the **Student Support Service**, as well as speaking to your lecturer or adviser.

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| | Ask: | ask.let@uea.ac.uk |
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