Bridging Between Algebra and Calculus

More Complicated Functions

This guide discusses how more complicated functions can be made from simpler functions. Particular attention is given to the composition of functions and how a function can be decomposed into simpler functions.

Introduction

Over the past 300 years the development of mathematical thinking has been closely associated with the evolution of the definition of a **function**. A function is a very important concept in mathematics, especially in calculus. Functions are also important in many subjects in which mathematics plays a crucial role such as computer science, economics and so on. If you are not familiar with how functions work and the way they are described you should read the study guides: *Functions* and *Using Functions* before continuing with this guide.

Often in mathematics you encounter complicated functions (in the form of expressions and equations). Understanding how such functions are made is very useful. For example it may help you to transpose a particularly intricate equation, correctly perform a calculation or to choose the most appropriate rule of differentiation or integration.

Many complicated functions are the result of a combination of other, simpler functions. The simpler functions usually comprise a basis of five basic functions which are used as the building blocks to create more complicated functions. These simple functions are specifically x^n, $\sin x$, $\cos x$, e^x and $\ln x$ (see the factsheet: *Five Basic Functions* for more details).

Using arithmetical operations to make new functions

One of the most common ways of making new functions is by taking the simple functions described above and combining them using the arithmetical operations (adding, subtracting, multiplying and dividing). You may undertake a combination of any number of these operations to make a new function.

Take the simple functions $f(x)=\cos x$ and $g(x)=x^3$, you can create new functions by:

1. Adding or subtracting them. $f(x)+g(x)=\cos x+x^3$

   $f(x)-g(x)=\cos x-x^3$

2. Multiplying them (**product**). $f(x)\cdot g(x)=\cos x\cdot\left(x^3\right)=x^3\cos x$

3. Dividing them (**quotient**) $\dfrac{f(x)}{g(x)}=\dfrac{\cos x}{x^3}$

   or $\dfrac{g(x)}{f(x)}=\dfrac{x^3}{\cos x}$

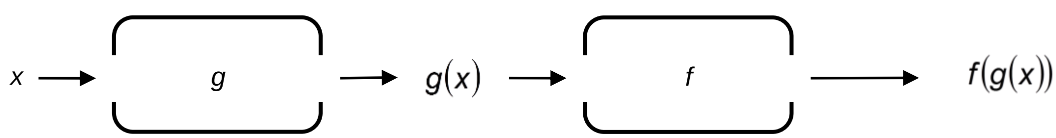
   Note the denominator cannot be zero.

4. Multiplying by a constant, *a*: $a\cdot f(x)=a\cos x$

   $a\cdot g(x)=ax^3$

Composition of functions

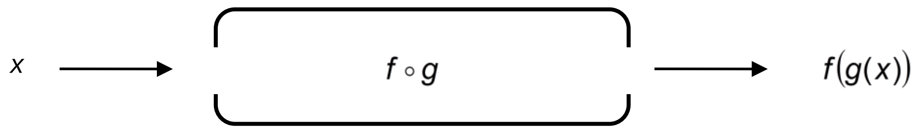
There is another, very common way, of producing more complicated functions, it is called **composition**. Functions made in this way are called **composite functions**. To make a composite function it is useful to think of the analogy of a function being a machine with an input and an output (see study guide: *Functions*). Composite functions result when a function acts as an input to another function. Take $f(x)$ and $g(x)$ from the previous example, if $g(x)$ is used as the input to $f(x)$ it could be pictured as:



In this diagram *x* is the input to the first machine (representing the function *g*). The output of this machine is $g(x)$ which acts as the input to the second machine (representing the function *f*). The output of the second machine is $f(g(x))$ as the function *f* is acting upon $g(x)$. The symbols $f(g(x))$, which also represent a function, are said "*f* of *g* of *x*". Looking at the machine above the first function acting on *x* is *g*, followed by *f* but if you read $f(g(x))$ left-to right it seems to imply that *f* happens first. When reading functions you should be aware of this, **the order in which the functions act are from inside out with respect to the brackets**, or from right to left if you prefer. Returning to the example, as $f(x)=\cos x$ and $g(x)=x^3$ you can interpret this procedure

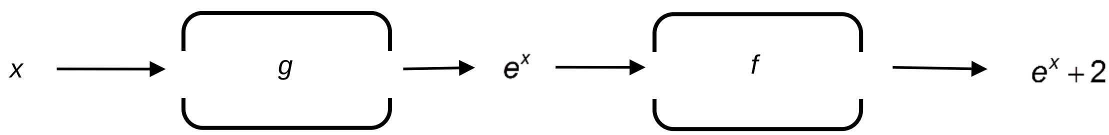
$f(g(x))$ as the original input x being cubed and then the cosine of x^3 being taken resulting in the composite function $f(g(x)) = \cos(x^3)$.

In $f(g(x))$ there are a lot of brackets, which may be confusing so mathematics uses the symbol $\circ$ to represent composition of functions. So $f(g(x))$ is written $f \circ g$ and similarly $g(f(x))$ is written $g \circ f$. In general $f \circ g$ **is not equal to** $g \circ f$. You can also picture $f \circ g$ as a single machine with an input x and an output $f(g(x))$:



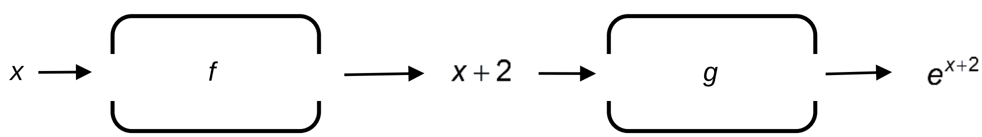
Example: What is the result of the compositions $f \circ g$ and $g \circ f$ if $f(x) = x + 2$ and $g(x) = e^x$?

The function *f* takes the input and adds 2. For the composition $f \circ g$, e^x is the input to *f*, so represented as a series of machines:



So $(f \circ g)(x) = e^x + 2$.

The function *g* raises *e* to the power of the input. For the composition $g \circ f$, $x + 2$ is the input to *g* so represented as a series of machines:



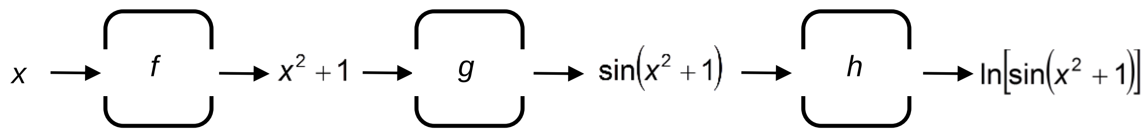
So $(g \circ f)(x) = e^{x+2}$, which is not the same as $(f \circ g)(x) = e^x + 2$. In $(f \circ g)(x)$ x acts as the power of *e* and then two is added to e^x. Whereas in $(g \circ f)(x)$, two is added to *x* and then $x + 2$ serves as the power of *e*.

Decomposing a function

You have seen how to create composite functions by using one function as an input to another. Being able to look at a complicated function and decide how it was made is an equally important skill known as **decomposition**. This can be challenging as decomposing a function is a skill acquired with experience. A good strategy is to look for the five basic functions (highlighted in the introduction) within the composite function and work out how they are combined. If there are brackets in the composite function you can use them to help you. Using BODMAS to help you assess the order and importance of operations, you can start from the innermost bracket and work outwards to decompose functions (see study guide: *What Do I Do First? BODMAS*).

Example: Decompose the composite function $y = \ln(\sin(x^2 + 1))$.

The innermost bracket in this function contains $x^2 + 1$ (a polynomial) which acts as the input to the sine function. The result of this acts as the input to the logarithm function. You can think of the overall process as a sequence of three function machines, the first *f* squares the input and then adds 1 or $f(x) = x^2 + 1$, the second *g* takes the sine of an input or $g(x) = \sin x$ and the third *h* takes the logarithm of an input or $h(x) = \ln x$. Given that *x* is your original input you can visualise the composition as the machines:



So the overall function $y = \ln(\sin(x^2 + 1))$ the composition of the functions *f*, *g* and *h* so:

$$y = (h \circ g \circ f)(x) = h(g(f(x))) = \ln(\sin(x^2 + 1))$$

Want to know more?

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