Learning Enhancement Team

University of East Anglia

STUDENT SUPPORT SERVICE

Bridging Between Algebra and Calculus

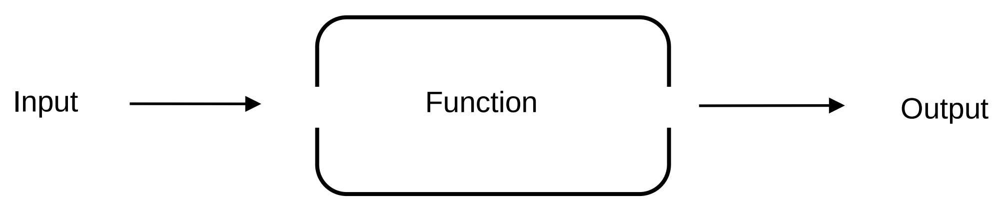
Using Functions

This guide introduces the concept of a function and its associated terminology. It explains domain, co-domain and range. It also describes how to use functions with inputs that are either numerical or algebraic.

Introduction

Over the past 300 years the development of mathematical thinking has been closely associated with the evolution of the definition of a **function**. A function is a very important concept in mathematics, especially in calculus. Functions are also important in many subjects in which mathematics plays a crucial role such as computer science, economics and so on. Having a good idea about the different types of numbers which exist in mathematics and the symbols used to describe them is beneficial when thinking about functions (see study guide: *Different Kinds of Numbers*).

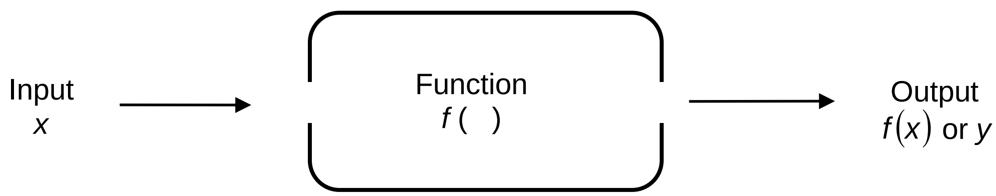
You can think of a function as a machine with an **input** and an **output**. The function itself controls how the input changes into the output.



The input can be a number, a letter or an algebraic expression. The function comprises mathematical operations or abstract rules which act upon the input and result in the output. For example, in economics the input of the machine may be the quantity of products sold and the output may be company profit, the function will control how much profit is made for a given amount of product sold. For example, in mathematics, the input may be x and the function could be “double the input and then add 3” giving the output $2x+3$.

Often the function itself is represented by a letter which embodies the machinery of the function. Commonly the letter f is used to describe a function as it is the first letter of the word function. The input of a function is more properly called the **independent variable** or **argument of the function** and is often represented in mathematics by the letter *x*. The output is more properly called the **dependent variable** or sometimes the **value of the function** and is often represented by the letter *y*. It is also common to represent the output by the notation $f(x)$ - which is said "***f* of *x***" or "**the value of *f* at *x***". It is important that you understand that *y* and $f(x)$ can be used interchangeably. However there are cases where using *y* is preferable to using $f(x)$ and *vice-versa.* For example if it is important that you know what the independent variable is in a function then $f(x)$ is preferable, as the input (in this case *x*) appears in brackets. You will often see *y* or $f(x)$ used to label the vertical axis of a graph.

You can think of a function *f* as having an effect on the input variable *x* to produce something new. In terms of the machine describe above:



Common Error

It is common to confuse the brackets in the notation $f(x)$ with multiplication or factorisation. Remember that the notation is used in functions to represent "the function *f* is evaluated at *x".* You may see example of this use of brackets in a variety of familiar mathematics such as $\sin(x)$ and $\ln(x)$. In contrast, in the expression $3x(x-2)$ the brackets indicate multiplication. This is an example of ambiguity in mathematical notation. You only know that $f(x)$ does **not** mean "multiply *f* by *x"* from the context. You should ensure you are comfortable using brackets in their proper context: a **Learning Enhancement Tutor** can help you with this.

In mathematics you will commonly see the letters *x*, *y* and *f* used to describe functions. In other disciplines other, more intuitive letters are used to represent variables and functions. For example $P(t)$ may represent profit *P* as a function of time *t* and $V(r)$ may represent volume *V* as a function of radius *r*. It is also common for a function to have more than one variable as an input; this is known as a **multivariable function**. For example if the area of a shape *A* is dependent on both its width *w* and height *h* you could write $A(w,h)$.

Domain, co-domain and range of a function

The set of valid inputs of a function have a special name, the **domain of the function**. The set of all the outputs associated with the domain is called the **range of the function** and the range itself is part of a bigger set called the **co-domain**. One important part of the definition of a function is that **for each individual element of its domain a unique output exists**. This is a difficult concept which is best illustrated by an example.

Example: What are the range and domain the function $f(x) = x^2$ where x are the integers from -2 to 2.

You can think of the **domain** of this function as the numbers -2, -1, 0, 1 and 2, which are the integers from -2 to 2 and so are potential values for x. You can think of the function f as a machine which takes one of these numbers and squares it; this is written as $f(x) = x^2$. The result is part of the **range** of the function. Take the number 1 as the input of the function, you could say that $x = 1$. You can calculate the corresponding output by substituting 1 into the function everywhere you see x. So:

$$\begin{array}{ccc} f(x) & = & x^2 \\ \downarrow & & \downarrow \\ f(1) & = & 1^2 = 1 \end{array}$$

which says that the function applied to the input 1 gives an output of 1. You can also say that 1 is part of the range. Following this method for each member of the domain you can see that:

when $x = -2$ $\quad f(-2) = (-2)^2 = 4$

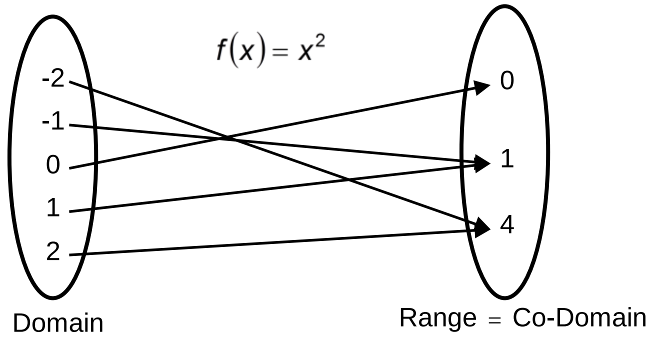
when $x = -1$ $\quad f(-1) = (-1)^2 = 1$

when $x = 0$ $\quad f(0) = 0^2 = 0$

when $x = 2$ $\quad f(2) = 2^2 = 4$

You have only three possible output values 0, 1, and 4 which make up the range of the function. The co-domain is *any* collection of objects which contains the range. For example the real numbers and the whole numbers are co-domains of this function.

There is a neat, pictorial way of depicting this concept:



In the above diagram, the domain is on the left and the co-domain is on the right with the arrows that connect them representing the function. The range comprises numbers in the co-domain at the end of an arrow. Here the range and co-domain are identical. These diagrams are discussed in more detail in the study guide: *Functions*.

Functions with different types of inputs

As you have seen, a function works like a machine which takes an input and transforms it to an output. The inputs are often, but not necessarily, numerical. The method of calculating output values (and hence the range) of a function with numerical inputs was touched upon in the example $f(x) = x^2$ above. Remember: in order to calculate the output corresponding to a specific numerical input you have to **substitute** the input value into the function everywhere you see the independent variable (usually denoted as *x*) and carry out the mathematical operations defined by the function.

Example: What is the value of the function $f(x) = 2x^2 - 3x + 1$ when $x = -2$?

The input value is -2 and so the question is asking you to calculate $f(-2)$. Substituting *x* with -2 in $f(x) = 2x^2 - 3x + 1$ gives:

$$f(-2) = 2 \cdot (-2)^2 - 3 \cdot (-2) + 1 = 2 \cdot 4 + 6 + 1 = 8 + 6 + 1 = 15$$

so $f(-2) = 15$. Take care when dealing with the negative sign. You can introduce brackets into a calculation and follow the BODMAS system (see study guide: *What Do I Do First? BODMAS*).

The input to a function does not have to be numerical; it can be a letter or even an algebraic expression. A function is simply a set of rules which transforms an input into an output. The function in the previous example $f(x) = 2x^2 - 3x + 1$ can be read as “square the input and multiply the result by two, subtract three times the input and then add 1”. The process of calculating the output of a function corresponding to a non-numerical input is identical to when you have a numerical input: **substitute** the input into the function everywhere you see the independent variable (usually denoted as *x*) and carry out the mathematical operations defined by the function.

Example: Given that $f(x) = 2x^2 - 3x + 1$, what is $f(p)$?

The procedure is the same as in the previous example but as the input is now *p* you to replace *x* with *p*:

$$f(x) = 2x^2 - 3x + 1$$
$$f(p) = 2p^2 - 3p + 1$$

Example: Given that $s(t) = 2t^2 - 3t - 5$, what is $s(h+1)$?

In this example the name of the function is *s* and the independent variable is *t*, you can say “*s* of *t* ”. To calculate $s(h+1)$, replace the instances of *t* in the function with the new input $h+1$:

$$s(t) = 2t^2 - 3t - 5$$
$$s(h+1) = 2(h+1)^2 - 3(h+1) - 5 \quad .$$

This leaves you with some algebra to perform, if you find the following algebra difficult you can read the study guides: *Steps into Algebra* to help you. So:

$$\begin{aligned} s(h+1) &= 2(h^2 + 2h + 1) - 3(h+1) - 5 \\ &= 2h^2 + 4h + 2 - 3h - 3 - 5 \\ &= 2h^2 + h - 6 \end{aligned}$$

Example: Given that $g(\theta) = \cos(\theta)$, what is $g(x^2 + 3)$?

Here the input θ has been replaced with $x^2 + 3$ as the input to cosine and so:

$$g(x^2 + 3) = \cos(x^2 + 3)$$

When the input to a function is another function the output is known as a **composite function** (for more details see study guide: *More Complicated Functions*).

Example: What is the value of the multivariable function $f(x,z) = x^2 + xz + z^2$ when $x = 2$ and $z = -3$?

To evaluate this you must substitute 2 for every appearance of *x* and -3 for every appearance of *z* in the function $f(x,z)$. So:

$$f(x,z) = x^2 + xz + z^2$$

$$f(2,-3) = 2^2 + 2\cdot(-3) + (-3)^2 = 4 - 6 + 9 = 7$$

So $f(2,-3) = 7$.

Want to know more?

If you have any further questions about this topic you can make an appointment to see a **Learning Enhancement Tutor** in the **Student Support Service**, as well as speaking to your lecturer or adviser.

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