Learning Enhancement Team

University of East Anglia

STUDENT SUPPORT SERVICE

Bridging Between Algebra and Calculus

Functions

This guide introduces some important concepts of functions: the domain, co-domain and range of a function. It shows you how to decide what is and what is not a function and then how to categorise functions. Finally it discusses which functions can be inverted.

Introduction

Functions are a fundamental building block of higher mathematics and are not just the common mathematical functions like x^2 or sine but are also found in databases and symbolic logic. They can also be used to describe very simple situations. For example, in a shop artichokes cost £1, broccoli costs £2 and cabbages also cost £2. Each vegetable has one price associated with it. In this study guide you will see that a function is a way to associate each **element** in the **set** of vegetables (artichoke, broccoli and cabbage) with an **element** in the **set** of prices (£1, £2, £3). By using information about these two sets, this guide will show you how to categorise functions. Not every price has a unique vegetable associated with it – a £2 vegetable might be broccoli or cabbage and not every price has a vegetable – no vegetable costs £3. This sort of information allows us to categorise the function and show that there is no inverse function in this case.

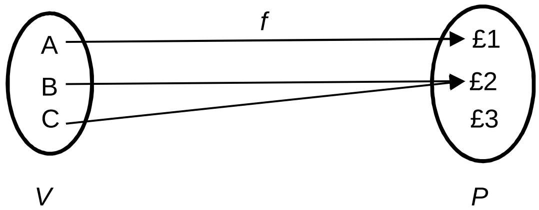
Domain, co-domain and range of a function

In mathematics, a **function** can be thought of as a **machine** which takes an **input** (often called x) and gives a **unique output** (often called $f(x)$, said "f of x"). In order to use functions well, you should consider what sort of things can be inputs and what sort of things can be outputs. For example the square root function $f(x) = \sqrt{x}$ cannot accept negative numbers or, in the grocer's example above, no vegetable can cost £1.50. The set of inputs for a given function have a special name, the **domain of the function**. The set of all the outputs associated with the domain is called the **range of the function** the range itself is part of a bigger set called the **co-domain**. Sets which are commonly used as domains, co-domains and ranges are the set of real numbers and the set of integers (see study guide: *Different Types of Numbers*). But a set can be any collection of objects and not just numbers. Sets are written as a letter (usually uppercase) or as a list of elements separated by commas in curly brackets $\{\}$. For example, in the grocer's example in the introduction, there are two sets: *V* which contains the vegetables and *P* which contains the prices. These are written as:

$$V = \{\text{Artichoke,Broccoli,Cabbage}\}$$

$$P = \{£1,\ £2,\ £3\}$$

A function takes an input which is an element in the domain *V* and connects it to an output which is an element in the range *P*. A function might also be a mathematical procedure (see study guide: *Using Functions*). In simple cases, there is a neat way of visualising them:



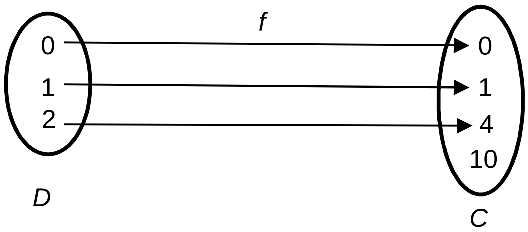
In the above diagram, the domain is on the left and the co-domain is on the right with the arrows that connect them representing the function. The range comprises numbers on the right at which an arrow ends. Here *V* is the domain and *P* is the co-domain. As no vegetable costs £3, the range is the set $\{1,\ 2\}$ which is part (a subset) of the co-domain. In other words 3 is in the co-domain but not in the range. You can write the function mathematically as:

$$f : V \to P$$

which can be read as "*f* connects *V* to *P*".

Example: *D* is the set of numbers $\{0,1,\ 2\}$, *C* is the set of numbers $\{0,1,4,10\}$ and the function $f(x) = x^2$ connects *D* to *C* (so $f : D \to C$). What is the range of *f* ?

Since $f(0) = 0^2 = 0$, $f(1) = 1^2 = 1$ and $f(2) = 2^2 = 4$ the function is shown as:



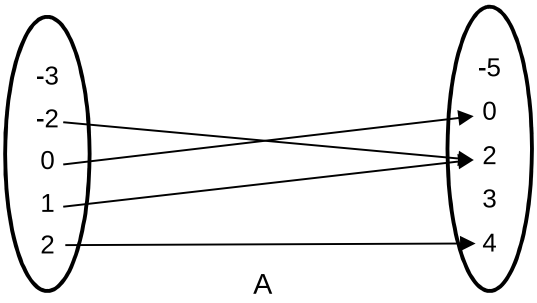
So *D* is the domain and *C* is the co-domain but, since no number in *D* goes to the number 10 in *C,* the range is the set $\{0,1,4\}$.

Identifying functions

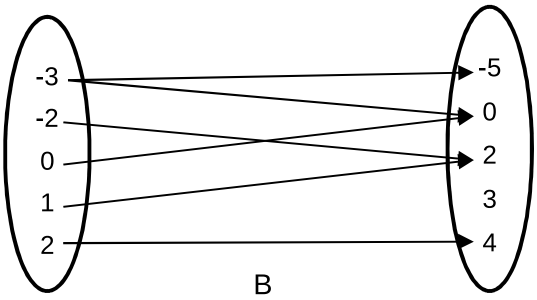
A function is like a machine: for every input there is a unique output. Think of a vending machine. Every code you put in gives an output, even if the output is “invalid code” or “out of stock”. Furthermore, if you type “245” into the vending machine you get out a specific chocolate bar. The output has to be unique. You do not get a chocolate bar some of the time and bottled water at other times. Mathematically, a function cannot have many outputs for a given input. Some connections between sets are not functions, either because there is no output for every input or because there is more than one output (this is known as a **one-to-many** relationship)

A function connects every element of its domain to only one element of the co-domain.

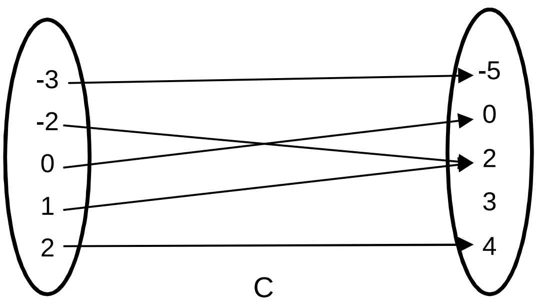
Example: Which of the following diagrams A, B, C represents a function?



Not a Function
The number -3 on the left has no connection to any value on the right.



Not a Function
The number -3 on the left has two connections -5 and 0 on the right. This is a one-to-many relationship.



Is a Function
Diagram C represents a function as each number in its domain has a unique corresponding value in the co-domain. It does not matter the 3 is not in the range or that both -2 and 1 on the left go to 2 on the right.

Types of functions

In the grocer's example, if a vegetable costs £2 can you tell what vegetable it is? Is there a vegetable for every price in P? In harder problems it is helpful to categorise functions to answer such questions. There are four categories of function:

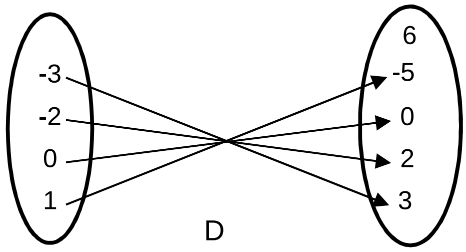
(i) When there is a unique connection between every element of the range and an element of the domain, the function is called **injective**, or an **injection**. An injection is also called a **one-to-one** function as one element in the range connects to only one value in the domain.

(ii) When the range of a function is equal to the co-domain, the function is called a **surjective function** or a **surjection**. Sometimes a surjection is called an **onto function** as its domain maps *onto* the whole of its co-domain.

(iii) A function which is both **surjective** and **injective** is called **bijective.** This is sometimes called **one-to-one and onto**.

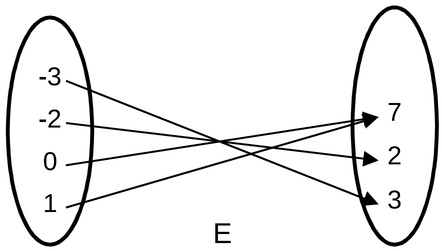
(iv) A function may be none of the above.

Example: Classify the functions depicted by diagrams D, E, F and G.



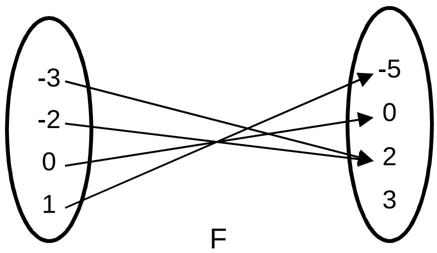
D

Diagram D is an **injection** because, for each value in the range there is a connection to only one value in the domain. However, it is not a surjection (and so not a bijection) because the value 6 of the co-domain is not connected to any value in the domain.



E

Diagram E is a **surjection** because its range is equal to its co-domain. However, it is not an injection (and so not a bijection) because 0 and 1 in the domain have the same value in the range, 7.



F

Diagram F is not a surjection because the value 3 in the co-domain has no connection with the domain. It is not an injection because -2 and -3 in the domain have the same value of 2 in the co-domain. So it cannot be a bijection.

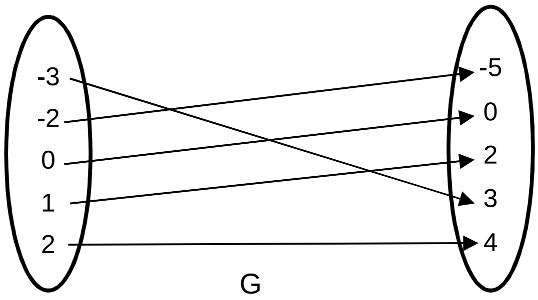


Diagram G is a **surjection** because its range is equal to its co-domain. It is an **injection** because each value of the range is paired with only one value of the domain. Therefore it is a **bijection**.

Bijections are invertible functions

The study guide: *Inverse Functions and Graphs* explains that an inverse function "undoes" what that function does. A function connects every member of its domain to the output which is a member of its co-domain. Then an inverse function must undo this. It must take every member of the original function's co-domain back to a unique member of its domain. **Not every function is invertible**. In fact a function is invertible if and only if every member of the co-domain has a unique connection with every member of the domain. In other words:

A function has an inverse if and only if it is a bijection.

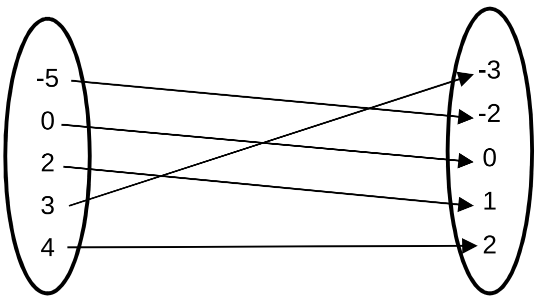
Example: Decide whether the functions in diagrams D, E, F and G have an inverse.

D is not a surjection and therefore not a bijection and therefore **does not have an inverse**. Specifically, there is no rule to map the -6 element of the co-domain onto the domain.

E is not a bijection and therefore **does not have an inverse**. Two numbers (0 and 1) map onto 7. If there was an inverse that maps back from the co-domain then would 7 go to 0 or to 1? This is a one-to-many relationship and therefore not a function.

F is not a bijection and therefore **does not have an inverse**.

G is a bijection and therefore **has an inverse**. Every element in the co-domain has a unique element in the range that it can map back to. The inverse would look like this:



Summary

		Type	Example
Not Functions		one-to-many	$f:\{A\}\rightarrow\{1,2\}$ $A\rightarrow 1 \quad A\rightarrow 2$
Functions		injection or one-to-one	$f:\{A,B\}\rightarrow\{1,2,3\}$ $A\rightarrow 1 \quad B\rightarrow 2$
Functions		surjection or onto	$f:\{A,B,C\}\rightarrow\{1,2\}$ $A\rightarrow 1 \quad B\rightarrow 2 \quad C\rightarrow 1$
Functions	Invertible Functions	bijection or one-to-one and onto	$f:\{A,B,C\}\rightarrow\{1,2,3\}$ $A\rightarrow 1 \quad B\rightarrow 2 \quad C\rightarrow 3$

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