# Learning Enhancement Team

University of East Anglia

STUDENT SUPPORT SERVICE

### *Steps into Trigonometry*

# Solving Trigonometric Equations

***This guide describes how to solve trigonometric equations. Such equations may have more than one solution. A good way to find the solutions is to draw a graph of the relevant trigonometric function.***

## Introduction

This study guide concerns solving **trigonometric equations** such as:

$$\cos x = 0.1$$

The study guide: *Trigonometric Ratios: Sine, Cosine and Tangent* describes how to solve such equations by using the $\cos^{-1}$ button on your calculator to find the angle *x*:

$$x = \cos^{-1} 0.1 = 84.36°$$

However this is not the whole answer. A trigonometric equation may have many solutions and your calculator will only give you one. The answer the calculator gives might not even be the one you want. You will have to understand more about the graphs of trigonometric functions in order to solve these equations.

In fact the equation $\cos x = 0.1$, where *x* can be any number, has infinitely many solutions. This is because trigonometric functions are not bijections and therefore do not have inverses, which of the infinitely many *x* values would the inverse function give? See the study guides: *Functions* and *Inverse Functions and Graphs* for more on this.

But there are **inverse trigonometric functions** which will solve trigonometric equations where the angle is restricted. It is important to realise that the values your calculator gives are not all the possible angles but only a range of angles as described below.

| | |
|---|---|
| When solving $\sin x = a$ | $-90° \le \sin^{-1} a \le 90°$ |
| When solving $\cos x = a$ | $0° \le \cos^{-1} a \le 180°$ |
| When solving $\tan x = a$ | $-90° \le \tan^{-1} a \le 90°$ |

Even though the inverse functions on your calculator only give answers in these ranges, some trigonometric equations will have solutions outside these ranges and so you will have to take care in finding these solutions.

## Method for solving trigonometric equations

All trigonometric functions are periodic (they repeat themselves) and give values for any value of $x$. Therefore if you are given the value of a trigonometric equation such as:

$$\cos x = 0.1$$

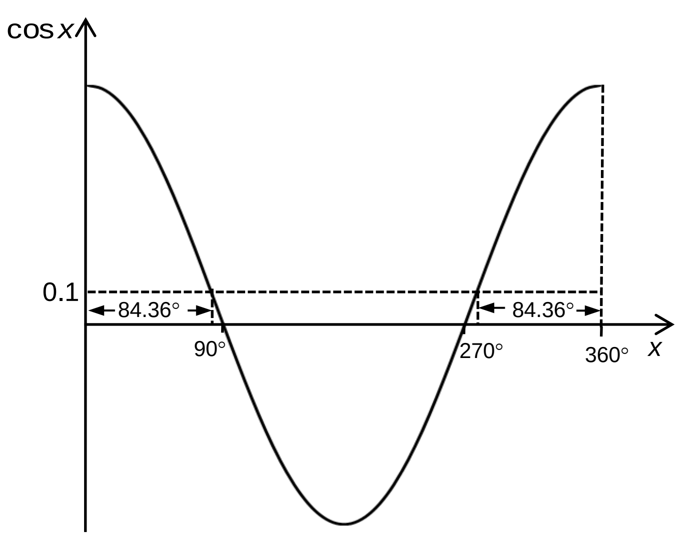
there are infinitely many angles $x$ which will satisfy it. Often a range of values known as an **interval**, is specified for $x$ meaning you will be expected to find all the solutions which lie in this interval. In order to solve this equation, it is very helpful to sketch the graph of the trigonometric function you are interested in.

*Example*: Solve $\cos x = 0.1$ for $0° \le x \le 360°$.

By taking inverse cosine of both sides of this equation, you get:

$$x = \cos^{-1} 0.1$$

and a calculator gives this value as 84.36°. Is this the only solution in the interval $0° \le x \le 360°$? The best way to find out is to draw the graph of the cosine function for this interval.




The dotted line goes across at $\cos x = 0.1$ and crosses the cosine graph at two points. The first one is near 90° and is the one given by your calculator. So you can see that $\cos 84.36° = 0.1$. However the dotted line also crosses the graph at another point near 270°. This solution is not in the range that the calculator gives you and so you have to work it out yourself. By using the symmetry of the graph you can see that the distance from this other point to 360° is also 84.36° and so the other solution is

$360° - 84.36° = 275.64°$. You can check with your calculator that $\cos 275.64° = 0.1$ as well. So the equation $\cos x = 0.1$ has two solutions in the interval $0° \le x \le 360°$, $x = 84.36°$ and $x = 275.64°$.

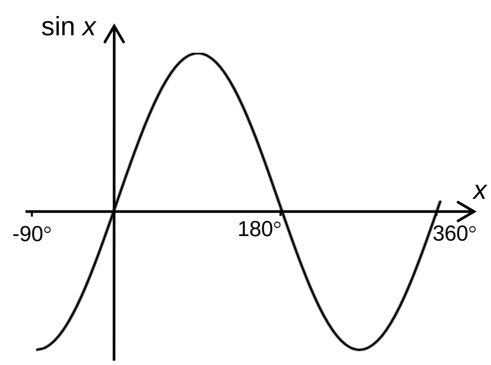
In general, the method for solving trigonometric equations is as follows:

1. Sketch the graph of the trigonometric function you are interested in.
2. Draw a line across the graph at the value required.
3. You can now see how many solutions there are by the number of intersections.
4. Use the inverse trigonometric function on your calculator.
5. Check the graph to see if this is a solution.
6. Compute the other solutions from the graph's symmetry.
7. Check your answers by putting them into the original equation.

The graphs of the trigonometric functions can be found in the factsheet: *Five Basic Functions.*

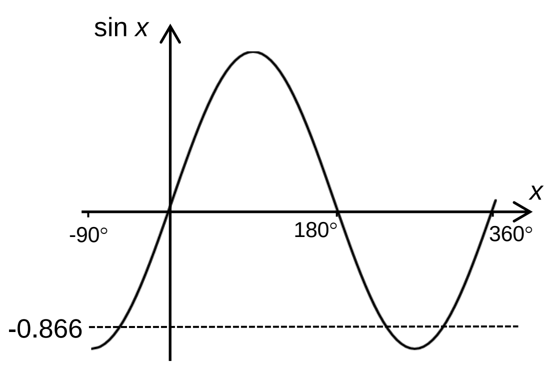
*Example*: Solve $\sin x = -0.866$ for $0° \le x \le 360°$.

1. Sketch a simple sine graph including the values in the required interval $0° \le x \le 360°$. Since the inverse sine function is defined from $-90°$ to $90°$ it is a good idea to include negative $x$ values up to $-90°$.




2. The dotted line is at $-0.866$.

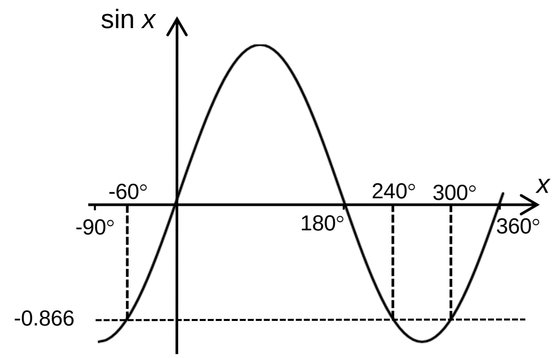
3. There are three intersections of the dotted line with the curve but only two of them lie in the required interval $0° \le x \le 360°$ and so there are two solutions.



4. The calculator gives $\sin^{-1}(-0.866) = -60°$. The range is given in degrees and so you must make sure your calculator is set to degrees and not radians.

5. This value is negative and so it is outside the required interval of $0° \le x \le 360°$.

6. Reading across the graph you can see that the first value in the range is more than 180°. In fact, looking at the graph, it is 60° more than 180° so it is 240°. The second value in the range is 60° less than 360° and so it is 300°. So the values of $\sin x = -0.866$ for $0° \le x \le 360°$ are 240° and 300°.




7. To check your answer, use a calculator to see that both $\sin 240° = -0.866$ and $\sin 300° = -0.866$.

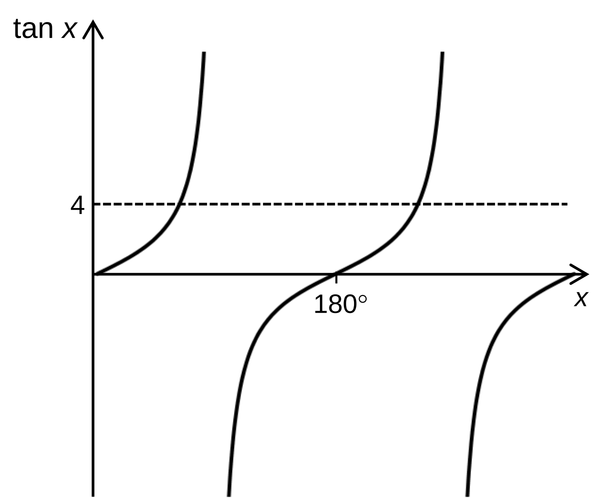
Sometimes the situation is more complicated. For example when the variable is transformed from *x* to 2*x* then other values may come into the given range.

*Example:* Solve $\tan 2x = 4$ for $0° \le x \le 180°$.

1. The graph of the tangent function is shown on the right.

2. The dotted line is at 4.

3. The dotted line crosses the graph in two places but one of them is outside the range $0° \le x \le 180°$. However, you will see that this value is still required. There are, in fact, two solutions.



4. The calculator gives $\tan^{-1} 4 = 76°$. So:

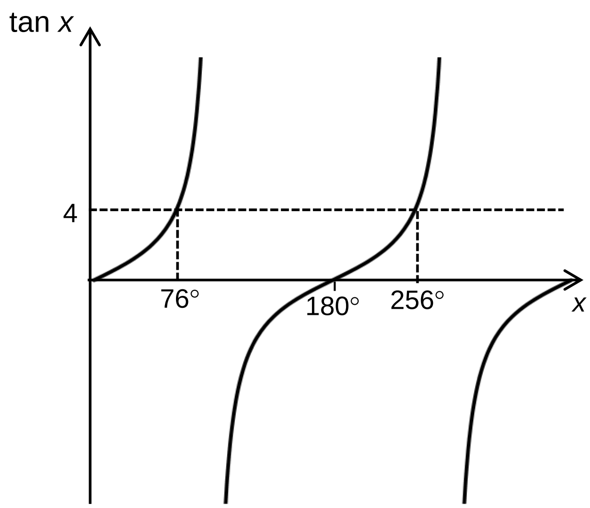
$$2x = \tan^{-1} 4 = 76°$$
$$x = 38°$$

5. This value is in the interval and so this in one answer.

6. From the graph, the other value of tangent equal to 4 is 76° more than 180°, which is 256°. This is outside the interval but remember that the question involves 2*x*, not *x*. So in fact $2x = 256°$ which means that:

$$x = 128°$$

which is inside the interval. So, the answers are $x = 38°$ and $x = 128°$.




7. To check, use a calculator to see that both $\tan(2 \times 38°) = 4$ and $\tan(2 \times 128°) = 4$.

You need to take extra care in examples like this, which involve a multiple such as 2*x*, to make sure you get all the solutions.

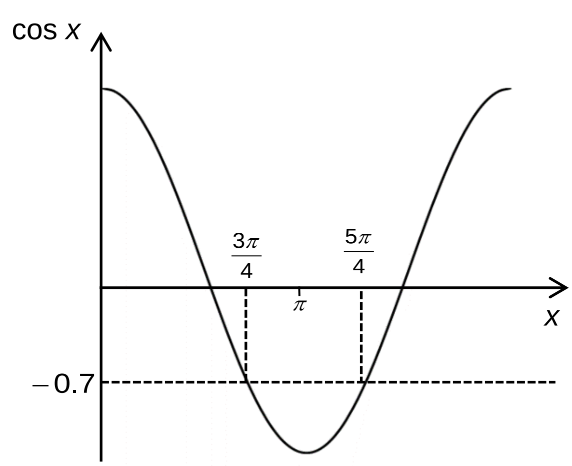
Another situation that can arise is when the variable is offset as in the next example.

*Example:* Solve $\cos(x + \pi) = -\dfrac{\sqrt{2}}{2}$ for $0 \le x \le \pi$ radians.

1. The cosine graph is shown on the right.

2. The dotted line is at $-\sqrt{2}/2$ which is approximately $-0.7$. It does not have to be accurate as it is just a sketch.

3. The dotted line crosses the cosine graph in two places but one of them is outside the range $0 \le x \le \pi$ radians. However you will see that this value is required.



4. The calculator gives $\cos^{-1}\left(-\sqrt{2}/2\right) = \frac{3}{4}\pi$ radians. The range is given in **radians** and so you must **make sure your calculator is set to radians and not degrees**. Then you have:

$$x + \pi = \frac{3}{4}\pi \text{ radians} \qquad \text{and so} \qquad x = -\frac{1}{4}\pi \text{ radians}$$

5. This value is outside the interval and so it is not required.

6. From the graph, the other value of cosine which equals $-\sqrt{2}/2$ is $\frac{3}{4}\pi$ radians less than $2\pi$ radians and so it is $\frac{5}{4}\pi$ radians which is outside the interval. However:

$$x + \pi = \tfrac{5}{4}\pi \text{ radians} \qquad \text{and so} \qquad x = \tfrac{1}{4}\pi \text{ radians}$$

which is inside the interval. As the question involves $x + \pi$ and not just $x$ you must take extra care to make sure you get the correct solution of $x = \frac{1}{4}\pi$ radians.

7. To check, use a calculator to see that $\cos\left(\frac{1}{4}\pi + \pi\right) = -\frac{\sqrt{2}}{2}$.

## Want to know more?

If you have any further questions about this topic you can make an appointment to see a **Learning Enhancement Tutor** in the **Student Support Service**, as well as speaking to your lecturer or adviser.

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