Learning Enhancement Team

University of East Anglia
STUDENT SUPPORT SERVICE

Steps into Trigonometry

The Cosine Rule

This guide introduces the cosine rule and illustrates the specific situations in when and how it can be used to solve problems involving triangles which do not contain a right-angle.

Introduction

Problems involving triangles are extremely common in many mathematical subject areas including geometry, vectors, calculus and complex numbers. Many such problems involve right-angled triangles and can be solved using the trigonometric ratios (see study guide: *Solving Right-Angled Triangles*). You should be comfortable solving right-angled triangles before continuing with this guide.

It is also common for problems to involve triangles which do not contain a right-angle. To solve these types of triangles you need to use the more general mathematical rules of either the **cosine rule** or the **sine rule**. You cannot use the trigonometric ratios (or Pythagoras' theorem) in these cases.

The table below shows all the triangles problems that are possible and the method you can use to solve them.

Type of Triangle	Known values	Method
Right-angled	Two sides	Trigonometric Ratios
Right-angled	One side and one angle	Trigonometric Ratios
Non right-angled	Three sides	Cosine Rule
Non right-angled	Two sides and included angle	Cosine Rule
Non right-angled	Two sides and non-included angle	Sine Rule
Non right-angled	One side and two angles	Sine Rule
Any	Three angles	Not Solvable

The cosine rule is given by:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

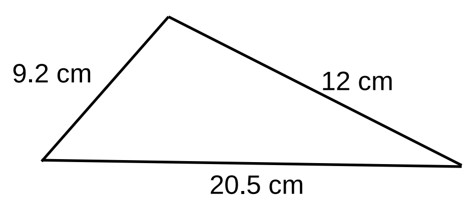
where *a*, *b* and *c* are the lengths of the sides of the triangle and *A* is the angle opposite *a*. The angle *A* is sometimes said to be **included** by *b* and *c*.

This study guide offers help with using the cosine rule. There is a separate guide for help with the sine rule, see study guide: *The Sine Rule*. To use the cosine rule effectively you should be comfortable with the manipulation of equations, see study guide: *Rearranging Equations* for help with this.

You can see from the table on the first page of this guide that you should use the cosine rule to help you solve a triangle if you know the lengths of all three sides *or* if you have the length of two sides and the size of the angle between them. As with all questions concerning triangles it is a good idea to sketch the triangle you are investigating and label the quantities you know as part of answering the question.

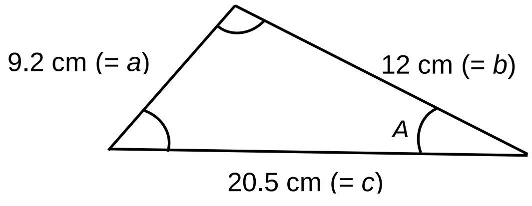
Example: (*Finding the size of angles using the cosine rule*)
Solve the triangle with sides 9.2 cm, 12 cm and 20.5 cm.

If you have the lengths of the three sides you can use the cosine rule to calculate **any** of the internal angles of the triangle. You can do this twice and then use the fact that the angles of a triangle add to 180° to find the third angle. Start by sketching the triangle (it does not matter if it is not accurate but you should try to get the longest and shortest sides to correspond with the biggest and smallest lengths):



As you have **all three sides** you can use the **cosine rule**. Follow this method:

(i) Decide on the angle you want to work out (it does not matter which) and label it *A*.
(ii) Label the side opposite this angle as *a* (**this is important**).
(iii) Label the other two sides *b* and *c* (it does not matter which way around).



Which means that $a = 9.2\,\text{cm}$, $b = 12\,\text{cm}$ and $c = 20.5\,\text{cm}$. Now substitute these values into the cosine rule given on the first page of this guide:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

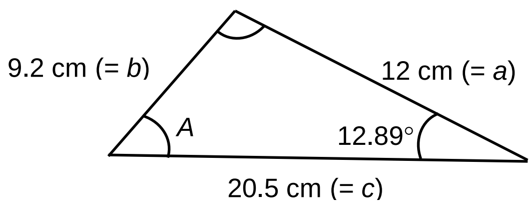
$$9.2^2 = 12^2 + 20.5^2 - 2\times 12\times 20.5\times \cos A$$

So: $$84.64 = 144 + 420.25 - 492\times \cos A$$

The only unknown quantity is *A* and so you can rearrange the final equation to give:

$$\cos A = \frac{84.64 - 144 - 420.25}{-492} = 0.97$$

And so $A = \cos^{-1} 0.97 = 12.89°$. You can repeat this procedure to find another angle. Remember you will label *a*, *b* and *c* differently because you are finding a different angle.



This time $a = 12$ cm, $b = 9.2$ cm and $c = 20.5$ cm. Now substitute these *new* values into the cosine rule given on the first page of this guide:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

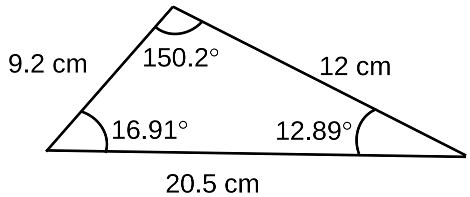
$$12^2 = 9.2^2 + 20.5^2 - 2\times 9.2\times 20.5\times \cos A$$

So $$144 = 84.64 + 420.25 - 377.2\times \cos A$$

Again, the only unknown quantity is *A* and you can rearrange the final equation to give:

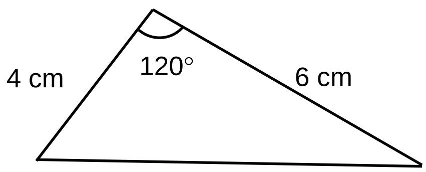
$$\cos A = \frac{144 - 84.64 - 420.25}{-377.2} = 0.96$$

And so $A = \cos^{-1} 0.96 = 16.91°$. You can use the fact that all the angles in a triangle add to 180° to show that the final unknown angle is $180° - 12.89° - 16.91° = 150.2°$ to solve the triangle as shown on the next page.



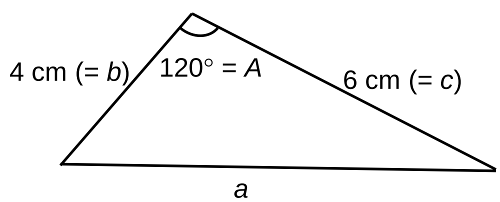
Example: (*Finding the length of a side*)
Solve the triangle with sides of length 4 cm and 6 cm which includes the angle 120°.

If you have the lengths of two sides and the angle included by them you can use the cosine rule to calculate the length of the third side. Again you start by sketching the triangle (it does not matter if it is not accurate but you should try to get the longest and shortest sides to correspond with the biggest and smallest lengths):



To find the length of the other side you can follow this method:

(i) Label the angle you know as A (**this is important**).
(ii) Label the side opposite this angle as a (**this is also important**).
(iii) Label the two sides you know as b and c (it does not matter which way around).



Which means that $A = 120°$, $b = 4\,\text{cm}$ and $c = 6\,\text{cm}$. Now substitute these values into the cosine rule given on the first page of this guide:

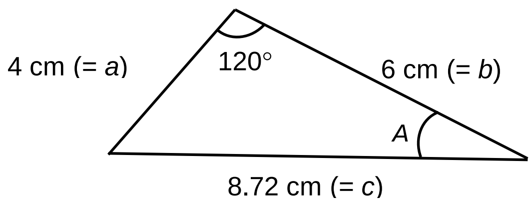
$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$a^2 = 4^2 + 6^2 - 2 \times 4 \times 6 \times \cos 120°$$

So $$a^2 = 16 + 36 + 24 = 76$$

By taking the square root of each side you find that $a = \sqrt{76} = 8.72$ cm to 2 decimal places.

As you now have all three sides you can find either of the unknown angles using the method from the previous example:

(i) Decide on the angle you want to work out (it does not matter which) and label it *A*.
(ii) Label the side opposite this angle as *a* (**this is important**).
(iii) Label the other two sides *b* and *c* (it does not matter which way around).



Which means that $a = 4$ cm, $b = 6$ cm and $c = 8.72$ cm. Now substitute these values into the cosine rule given on the first page of this guide:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$4^2 = 6^2 + 8.72^2 - 2 \times 6 \times 8.72 \times \cos A$$

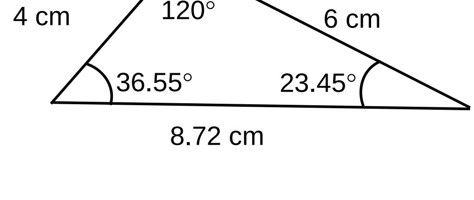
So $$16 = 36 + 76 - 104.64 \times \cos A$$

The only unknown quantity is *A* and so you can rearrange the final equation to give:

$$\cos A = \frac{16 - 36 - 76}{-104.64} = 0.92 \text{ to 2 decimal places}$$

And so $A = \cos^{-1} 0.92 = 23.45°$.

You can use the fact that all the angles in a triangle add to 180° to show that the final unknown angle is $180° - 120° - 23.45° = 36.55°$ to solve the triangle as shown below.



Want to know more?

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